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Injectivity conditions of rational Bézier surfaces

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ABSTRACT

Rational Bézier surface is a common fitting tool in Computer Aided Geometric Design. The injectivity of curve/surface implies the one-to-one property and there is no self-intersection of curve/surface. In this paper, we propose a geometric method for checking the injectivity of rational Bézier surface based on its control points.

1. Introduction

Injectivity of a function map plays an important role in image warping and morphing, 3D deformation, and volume morphing. It can avoid image fold, information loss and obtain desirable results. The injectivity of a curve/surface implies the one-to-one property and is equivalent to no self-intersection on the curve/surface. Hoffmann [5] pointed out that the intersection problem is a fundamental problem in the integration of geometric objects and solid modeling systems. Checking the possibility of the self-intersections and calculating the intersections of curves or surfaces are important in Computer Aided Geometric Design (CAGD).

In 1989, Lasser [3] proposed an algorithm to calculating the self-intersections of Bézier curves and spline curves. The "Angle criterion" proposed in his paper gave out that if the sum of the rotation angles is smaller than or equal to π, then the curve has no self-intersection. However, this is a rough estimation of the existence of the self-intersections since the curve may have no self-intersection though the sum of the rotation angles is bigger than π. We will explain the reason in this paper. Manocha and Canny [20], Laurent [21], and Patrikalakis [22] studied the intersections of curve to surface (or surface to surface). Their works focused on the calculation of the self-intersections not the checking method for the existence of self-intersections.

In 1994, Goodman and Unsworth [6] proposed the sufficient condition for the injectivity of 2D Bézier surface. Their condition contained $2m(m+1)+2n(n+1)$ linear inequalities for a $m \times n$ 2D tensor-product patch. Following this work, Hernández-Mederos et al. [23] proposed a sufficient condition for local injectivity of a 2D triangular cubic Bézier function in 2006. The locally sufficient injectivity conditions of 2D and 3D uniform cubic B-spline surfaces were proposed by Choi and Lee [7] and they applied the methods to image morphing. Floater [24] proved that convex combination mappings are one-to-one if the boundary of the triangulation is homeomorphic to a convex polygon. Recently, Müller et al. [19] gave the conditions of injectivity of the polynomial mappings based on the theory of algebraic geometry and combinatorics.

We consider the sufficient and necessary condition of the injectivity of rational Bézier surface. To study dynamical systems arising from chemical reaction networks, Craciun et al. [8] presented an injectivity theorem for certain mappings. Based on this theorem, Craciun et al. [9] proposed a geometric condition on control points of toric Bézier surface. This geometric condition is equivalent to the surface with no self-intersection for arbitrary choice of positive weights. However, the result in [9] only guarantees injectivity in the interior of a patch. Sottile and Zhu [10] corrected this minor flaw, at least for 2D patches. For 3D rational Bézier curve case, Zhu and Zhao [11] proposed the geometric condition on the control polygon which implies that rational Bézier curve has no self-intersection for any choice of positive weights. It is well known that the boundary curves of rational Bézier surface are rational Bézier curves. Since the interior of a surface may intersect with its boundary curves, the conditions of interior control points for checking the injectivity of the rational Bézier surface should be reconsidered. In this paper, we will present a geometric condition for control points of the rational Bézier surface which guarantees the injectivity of surface.

In fact, the shape of control net implies the shape of the surface. This phenomenon brings us a question: *Does no self-intersection of*

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0097-8493/

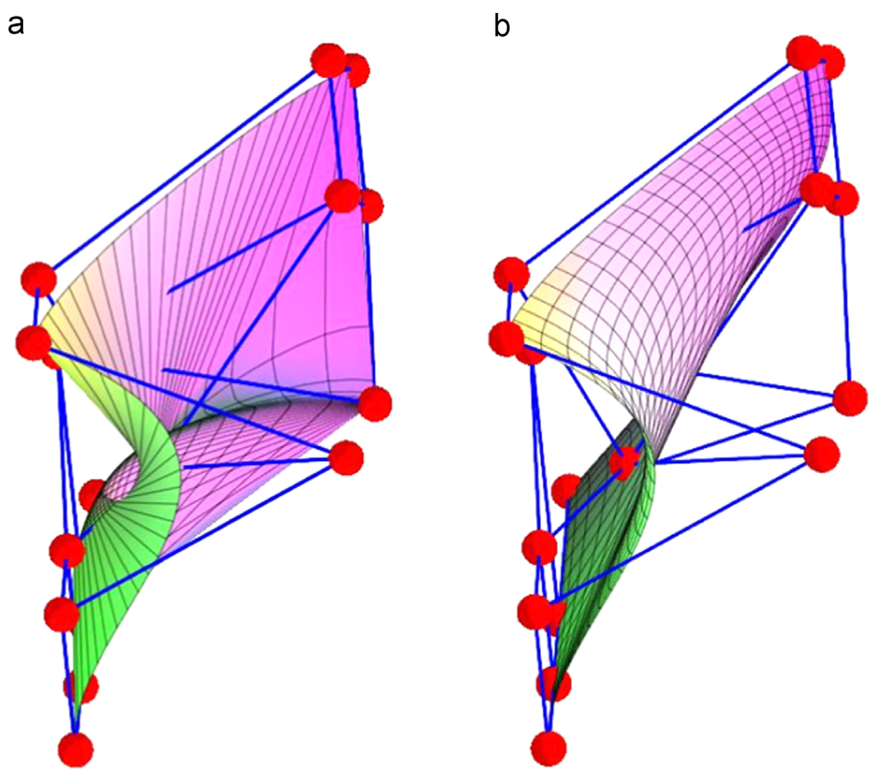

Fig. 1. Bicubic rational Bézier surfaces. (a) The bicubic rational Bézier surface with self-intersections and (b) the bicubic rational Bézier surface without self-intersections.

control net of rational Bézier surface imply no self-intersection of the surface? The answer is NO. For example, the control net in Fig. 1 has no self-intersection. It is obvious that surface with this control net in Fig. 1a has self-intersections, which can be removed by changing another set of positive weights as shown in Fig. 1b. We will explain the reason in Section 3.

In this paper, we define a geometric characteristic of the set of control points, called well-posedness, which implies the injectivity of the rational Bézier surfaces for any choice of positive weights. The description of some definitions is based on toric Bézier surfaces introduced by Krasauskas [12] and the proofs of our results are based on toric degenerations of Bézier surfaces proposed by García-Puente et al. [13]. We also indicate that when the control points lie on a plane, our result is equivalent to the injective condition for 2D rational Bézier surfaces presented by Sottile and Zhu [10]. We prove results of rational tensor product Bézier surfaces in Section 3. For rational Bézier triangle surface, we only list the results without proof.

2. Rational Bézier surface and toric degeneration

In this section, we employ the equivalent definition of rational Bézier surface derived from Krasauskas's toric surface patch [12]. In order to prove our main result, we recall the degree elevation algorithm (refer to [1,2,14]) and the toric degeneration theory presented by García-Puente et al. [13] of rational Bézier surface.

Definition 1. The parametric surface:

$$\mathbf{R}(u,v)=\frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\omega_{i,j}\mathbf{P}_{i,j}B_i^m(u)B_j^n(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n}\omega_{i,j}B_i^m(u)B_j^n(v)},\quad (u,v)\in[0,1]\times[0,1] \tag{1}$$

is called a *rational Bézier surface* of degree $m\times n$, where $B_i^m(u)$ and $B_j^n(v)$ are tensor-product Bernstein basis functions of degree $m\times n$, $\mathbf{P}_{i,j}$ are control points and $\omega_{i,j}$ are the weights. The control net $\mathcal{N}$ is the $m\times n$ grid composed of connecting the neighboring control points in the same row and column using line segments.

The rational Bézier surface $\mathbf{R}(u,v)$ can be represented as a rational Bézier surface of degree $(m+1)\times(n+1)$ by the degree elevation algorithm:

$$\mathbf{R}(u,v)=\frac{\sum_{i=0}^{m+1}\sum_{j=0}^{n+1}\omega^*_{i,j}\mathbf{P}^*_{i,j}B_i^{m+1}(u)B_j^{n+1}(v)}{\sum_{i=0}^{m+1}\sum_{j=0}^{n+1}\omega^*_{i,j}B_i^{m+1}(u)B_j^{n+1}(v)}, \tag{2}$$

where

$$\omega^*_{i,j}=\alpha_i\beta_j\omega_{i-1,j-1}+\alpha_i(1-\beta_j)\omega_{i-1,j}+(1-\alpha_i)\beta_j\omega_{i,j-1}+(1-\alpha_i)(1-\beta_j)\omega_{i,j}$$

$$\mathbf{P}^*_{i,j}=(\alpha_i\beta_j\omega_{i-1,j-1}\mathbf{P}_{i-1,j-1}+\alpha_i(1-\beta_j)\omega_{i-1,j}\mathbf{P}_{i-1,j}+(1-\alpha_i)\beta_j\omega_{i,j-1}\mathbf{P}_{i,j-1}+(1-\alpha_i)(1-\beta_j)\omega_{i,j}\mathbf{P}_{i,j})/\omega^*_{i,j},$$

where the coefficients $\alpha_i=i/(m+1),\beta_j=j/(n+1)$. The result from [1,2,14] shows that the limit of sequence of control nets obtained by degree elevation is the surface itself.

Rational Bézier surfaces are particular cases of toric surface patches (refer to [12]) with the spacial coefficients. We illustrate the definition following Krasauskas's toric patches, which is equivalent to its traditional definition.

Let $\mathcal{A}_{m,n}=\{(i,j)\,|\,i=0,1,\ldots,m;j=0,1,\ldots,n\}$, m,n be two positive integers, and $\square_{m,n}=\Delta_{\mathcal{A}_{m,n}}=conv\{\mathcal{A}_{m,n}\}$ be a lattice polygon with four vertices (0,0), $(m,0),(0,n)$ and (m,n). The boundary lines of $\square_{m,n}$ defined by $f_1(u,v)=u$, $f_2(u,v)=v$, $f_3(u,v)=m-u$ and $f_4(u,v)=n-v$.

Definition 2. Given $\mathcal{A}_{m,n}\subset\mathbb{Z}^2$, control points set $\mathcal{P}=\{\mathbf{P}_{i,j}\,|\,(i,j)\in\mathcal{A}_{m,n}\}\subset\mathbb{R}^3$ and weights $\omega=\{\omega_{i,j}>0\,|\,(i,j)\in\mathcal{A}_{m,n}\}$, the toric Bézier surface associated with rectangle $\square_{m,n}$ is parameterized by the rational