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Polynomial spline surfaces with rational linear transitions

Jörg Peters*, Martin Sarov

SurfLab CISE, University of Florida, Gainesville, FL 32611, USA

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ABSTRACT

We explore a class of polynomial tensor-product spline surfaces on 3-6 polyhedra, whose vertices have valence $n=3$ or $n=6$. This restriction makes it possible to exclusively use rational linear transition maps between the pieces: transitions between the bi-cubic tensor-product spline pieces are either C^1 or they are G^1 (tangent continuous) based on one single rational linear reparameterization. The simplicity of the transition functions yields simple formulas for a hierarchy of splines on subdivided 3-6 polyhedra.

1. Introduction

For genus $g=1$, e.g. the torus, an affine atlas can be constructed, for example, using B-splines on a tensor-product grid, made periodic by setting equal the opposing edges. For genus $g>1$, as a consequence of the Gauss–Bonnet Theorem, a closed, orientable, smooth 2-manifold does not have an affine atlas. However, a paper in this conference series [11] showed that there is a *rational linear reparameterization* for constructing smooth surfaces of genus $g>0$ from tensor-product splines. If the reparameterization is structurally symmetric, i.e. indexing does not matter (cf. Definition 4), then it is unique.

Together with Lemma 4 of [10], exclusive use of a structurally symmetric rational linear reparameterization implies that the resulting piecewise tensor-product spline surfaces must be assembled from basic pieces with the *restricted layout*: wherever the spline patches join with purely geometric continuity (G^1 continuity, not C^1 continuity) across a boundary, its end vertices must have equal valence.

By the "hairy ball" theorem (see e.g. [4]), a regular surface of genus $g>1$ must have transitions with purely geometric continuity. The paper [11] derived the apparently unique restricted layout for quad-only polyhedra: every quad has vertices of valence 8 or is a grid-like refinement thereof. This paper now presents a second, more natural family of restricted-layout quadrilateral polyhedra that uses less dramatic valences, namely any combination of $n=3$ and $n=6$ or a grid-like refinement of the quads (see Fig. 1).

The contributions of this paper are as follows:

- We exhibit a family of quadrilateral polyhedra, the 3-6 polyhedron, with vertices of valence $n=3$ or $n=6$, that satisfy the restricted layout. (Grid-like partition of the 3-6 quads of the 3-6 polyhedron yields additional vertices of valence $n=4$, see Figs. 1 and 2).
- The simplicity of the transition functions yields simple formulas for a hierarchy of splines on grid-like subdivided input meshes. Each finer level offers additional geometric degrees of freedom (see Fig. 3) for detailed models.

Overview: Section 2 reviews the pertinent literature. Section 3 formalizes the notion of a 3-6 polyhedron, its refinement and properties, the notion of structurally symmetric geometric continuity and explains the structure of the resulting surfaces. Section 4 introduces a hierarchy of G^1 spline surfaces on subdivided 3-6 polyhedra, by providing a concrete algorithm and proving geometric continuity.

2. Literature

Since regular, differentiable 2-manifolds (surfaces) of genus $g\geq 2$ without boundary do not have an affine atlas, we cannot hope to just make an affine change of variables to transition between all the tensor-product spline pieces of a regularly

* Corresponding author.

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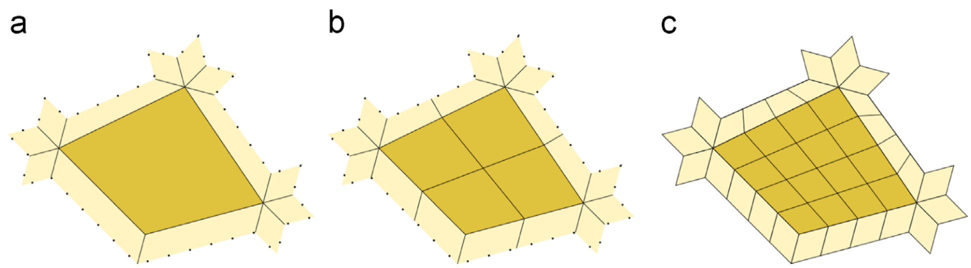

Fig. 1. (a) An example 3-6 quad with valences $n = 3, 6, 6, 6$ and its uniform binary subdivisions (b) at level $\ell = 1$ and (c) at level $\ell = 2$. Each (subdivided) 3-6 quad is associated with one tensor-product spline patch.

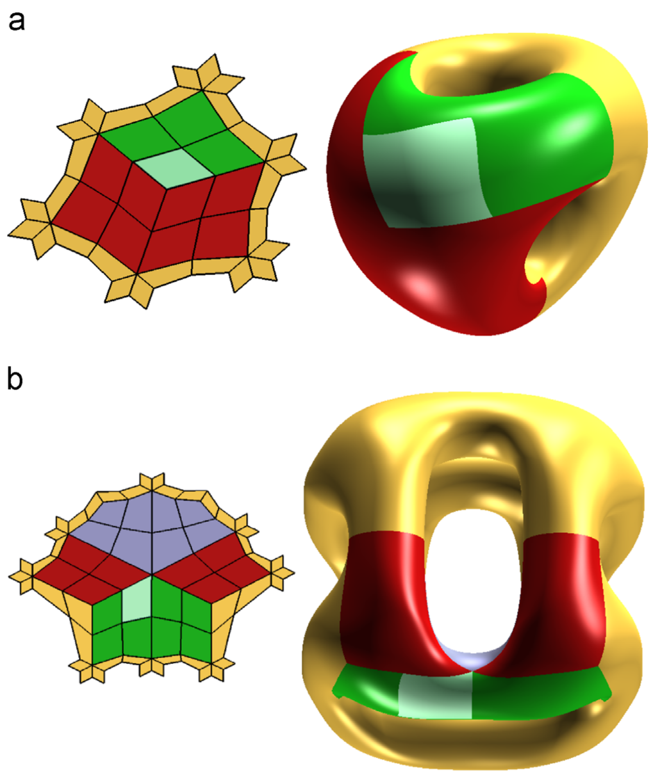

Fig. 2. The layout induced by once-subdivided 3-6 quad allows modeling smooth objects of any genus greater than one using one bi-cubic polynomial piece per sub-quad: (a) uses 2×6 quads, (b) 8×6 quads. (a) 3-neighborhood, level 1 [3,6,6,6] quads, genus $g = 2$ (b) 6-neighborhood, level 1 [3,6,6,6] quads, genus $g = 5$.

parameterized smooth closed surface of genus $g > 1$ in $\mathbb{R}^3$. However, if we allow rational linear reparameterization, we can, in principle, find conformal bijective orientation-preserving transition maps such as Möbius transformations. Concretely, Peters and Fan [11] exhibited a unique rational linear reparameterization that allows structurally symmetric construction of smooth surfaces of genus $g > 0$. This reparameterization relates the derivatives of two abutting spline patches p and q along their common curve by *linear scaling*, i.e.

$$\partial_2 p(t,0) + \partial_2 q(0,t) = \omega(t)\partial_1 p(t,0) \qquad \text{(G1)}$$

where $\omega(t)$ is linear.

An independent result, [10, Lemma 4], rules out general G^1 constructions with $\omega(t)$ linear everywhere with one exception: the derivatives can be everywhere related by (G1) if the endpoints $p(0,0)$ and $p(1,0)$ have the same valence – or if ω in (G1) is constant. In short, "linearly G^1-connected vertices must have the same valence".

More recently, Beccari et al. [3] introduced RAGS, a construction using *rational three-sided macro* patches connected by a rational linear reparameterization (whereas the four-sided patches in [11] are *polynomial*). This approach generalizes the work of Alfeld, Neamtu and Schumaker, who obtained functions on surfaces by restricting trivariate polynomials to a genus $g = 0$ surface [2] and credited part of their approach to [9]. Pottmann and Wallner applied hyperbolic geometry to model smooth surfaces of higher genus by rational splines: [13,12] used the classical Möbius map to assemble an atlas from overlapping disks. Atlas-based surface construction was pioneered by [5].

While Beccari et al. [3] propose a computation to enforce constraints that allow for a rational linear reparameterization, a careful reading of [3] shows that, in the *structurally symmetric* case, their construction uses the rational linear reparameterization derived in [11] for three-sided patches – an observation that the uniqueness of the parameterization up to a scalar degree of freedom already predicts. Once a structurally symmetric base atlas of three-sided patches has been created, each three-sided patch can be irregularly partitioned into triangular macro-patches and additional vertices resulting from splitting triangles can have any number of neighbors (see Fig. 4a). It is only after pruning away these extra triangulations that the underlying structure becomes apparent and shows that structurally symmetric G^1 constructions are equally restricted for triangular patches as for tensor-product patches.

3. Restricted quad-layouts compatible with rational linear transition maps

Below, in Section 3.1, we define the restricted patch layout and a class of polyhedra, the 3-6 polyhedra, that can be endowed with a piecewise tensor-product spline surface to satisfy a restricted patch layout. In Section 3.2 we review the definition and results concerning the geometric continuity of structurally symmetric rational linear transition maps. Finally, in Section 3.3, we show how we index the tensor-product patches and their coefficients.

3.1. Restricted quad-layout and 3-6 polyhedron

Euler's formula, $v - e + f = \chi$, characterizes closed polyhedra by relating the numbers of vertices v, edges e and faces f of a given polyhedron to the Euler characteristic (Euler–Poincaré characteristic) $\chi = 2 - 2g$. Here g is the topological genus, i.e. the number of handles. For example, Fig. 2b shows a surface of genus 5 built exclusively from quads whose vertices have valence $n = 3$, 6, 6 and 6. We note that since we can associate with each 4-valent vertex four half-edge and four quarter of attached quads the net contribution of 4-valent vertices to the Euler count is $1 - 4/2 + 4/4 = 0$. Then adding or removing checkerboard arrangements of quadrilaterals does not affect the left hand side $v - e + f$ of Euler's formula (cf. Fig. 4b).

Lemma 4 of [10] proves that structurally symmetric geometrically smooth surface constructions must relate some abutting patches with quadratic or higher-degree reparameterizations – unless they have a restricted layout. Restricted layout relates the