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A planar quadratic clipping method for computing a root of a polynomial in an interval

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ABSTRACT

This paper presents a new quadratic clipping method for computing a root of a polynomial $f(t)$ of degree n within an interval. Different from the traditional one in $\mathbb{R}^1$ space, it derives three quadratic curves in $\mathbb{R}^2$ space for approximating $(t, f(t))$ instead, which leads to a higher approximation order. Two bounding polynomials are then computed in $O(n^2)$ for bounding the roots of $f(t)$ within the interval. The new clipping method achieves a convergence rate of 4 for a single root, compared with that of 3 from traditional method using quadratic polynomial approximation in $\mathbb{R}^1$ space. When $f(t)$ is convex within the interval, the two bounding polynomials are able to be directly constructed without error estimation, which leads to computational complexity $O(n)$. Numerical examples show the approximation effect and efficiency of the new method. The method is particularly useful for the fly computation in many geometry processing and graphics rendering applications.

1. Introduction

Computing the solutions of (systems of) non-linear equations is frequently needed for solving various geometric and graphics problems [1,2,8,9,12,13,15,20,22],. Typical examples include curve/surface intersection and surface rendering by ray-tracing [7,15,18,19], point projection, which is to compute the closest point on a curve or surface to a given point [3], collision detection [4,14], and bisectors/medial axes computation [8]. The method is particularly useful for the fly computation in connection with the above applications in both geometry processing and graphics rendering.

For solving a polynomial equation, the Descartes rule [5,21] and the Sturm theorem [10,11] are two well-known techniques for isolating real roots. A collection of related references can be found in [16]. A Bézier curve is bounded by the convex hull of its control polygon, the corresponding roots are then bounded by the roots of the convex hull. Note that the number of the zeros of a Bézier function is less or equal to that of its control polygon, the zeros of its control polygon can be used for isolating the zeros of a Bézier function [17].

There are also several clipping algorithms developed for finding the roots of polynomials [1,15,17,23]. In 1990, Sederberg and Nishita proposed a Bézier clipping technique, which is to identify regions of the polynomial which do not include the part of the roots by using the convex hull property of Bézier curves [23]. The Bézier clipping method was proved to be quadratically convergent in [24].

Mørken and Reimers presented a linear approximation method [17], in which the given B-Spline (or Bézier) curve is approximated by its control polygon, which achieves a convergence rate of 2. Bartoň and Jüttler presented a quadratic clipping method for computing all the roots of a univariate polynomial equation within an interval [1]. In the quadratic clipping method in [1], the original univariate polynomial is firstly approximated by a quadratic polynomial based on degree reduction. Then two quadratic polynomials bounding the original polynomial within an interval are estimated, and the roots of the original polynomial are also bounded by the roots of the two bounding polynomials. The quadratic clipping method achieves a convergence rate of 3 for a single root and a super-linear rate of 3/2 for double roots, which is faster than the Bézier clipping method.

Later, Liu et al. extended the quadratic clipping technique and proposed a cubic clipping method, which uses a cubic polynomial to approximate the original polynomial based on degree reduction and achieves a convergence rate of 4 for single roots [15].

This paper presents a planar quadratic clipping method for computing a root of a polynomial within an interval, which can achieve an optimal convergence rate of 4 for a single root by using quadratic polynomial curves. Suppose that the given polynomial $f(t)$ is of degree n. Instead of approximating $f(t)$ in $\mathbb{R}^1$ space by using degree reduction, we use a planar quadratic polynomial curve to

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approximate the curve $(t,f(t))$ in $\mathbb{R}^2$ space. Three constructive interpolation methods QT_1, QT_2 and QT_3 are proposed, in which the quadratic curve interpolates either two points and their two directional tangent vectors, or three points and one directional tangent vector. QT_1 and QT_2 are used for convex cases that the curve $(t,f(t))$ has no inflexion point, while QT_3 is applied for other cases. In principle, QT_1 and QT_2 can be used for bounding $f(t)$ within a linear computation complexity, which is much better than that of previous clipping methods $O(n^2)$. Numerical examples illustrate the effectiveness and the effect of the new method.

The remainder of this paper is organized as follows. Section 2 introduces three constructive quadratic tangent methods with optimal approximation order that are used as the foundation of the proposed root finding method. Section 3 presents the new root finding method named planar quadratic clipping, while Section 4 discusses the computation complexity and convergence of the proposed method. Some further examples with comparisons are given in Section 5, and the conclusions are drawn at the end of this paper.

2. Constructive planar quadratic clipping methods

Given a smooth function $f(t)$ and an interval $[a,b]$. For the sake of convenience, we transfer the parameter interval $[a,b]$ into $[0,1]$ by using $\overline{f}(t)=f(a+ht)$ and $\overline{f}'(t)=f'(a+ht)h$, where $h=b-a$. Without loss of generality, suppose that the given interval is $[0,1]$. Let

$$\mathbf{p}_0=(0,\overline{f}(0))=(0,p_0),\quad \mathbf{p}_m=(0.5,\overline{f}(0.5))=(0.5,p_m),$$
$$\mathbf{p}_1=(1,\overline{f}(1))=(1,p_1),\quad \mathbf{v}_0=(1,\overline{f}'(0))=(1,d_0),$$
$$\mathbf{v}_m=(1,\overline{f}'(0.5))=(1,d_m),\quad \mathbf{v}_1=(1,\overline{f}'(1))=(1,d_1),$$
$$\lambda_0=\det(\mathbf{p}_1-\mathbf{p}_0,\mathbf{v}_0),\quad \lambda_1=\det(\mathbf{p}_1-\mathbf{p}_0,\mathbf{v}_1),$$

where "det" denotes determinant of the two vectors. Suppose that the quadratic Bézier curves are

$$\mathbf{A}_j(u)=(\overline{x}_j(u),\overline{y}_j(u))$$
$$=\mathbf{p}_0B_0^2(u)+\mathbf{q}_jB_1^2(u)+\mathbf{p}_1B_2^2(u),\quad j=1,2,3, \tag{1}$$

which are used to approximate the curve $\overline{\mathbf{C}}(t)=(t,\overline{f}(t))$ within the interval $[0,1]$, where $B_0^2(u)=(1-u)^2$, $B_1^2(u)=2(1-u)u$ and $B_2^2(u)=u^2$ are Bernstein polynomials.

2.1. Outline of the idea

In the previous quadratic clipping method, it utilizes a quadratic polynomial $y(t)$ to approximate $f(t)$, while in the planar quadratic clipping method, it utilizes a quadratic polynomial curve $(x_2(u),y_2(u))$ to approximate $(t,f(t))$ in $\mathbb{R}^2$ space instead, which needs to estimate the mapping function between parameters u and t. In principle, the planar quadratic clipping method is to find a quadratic reparameterization function $\phi(t)=t+\rho_1(t-1)t$ and a quadratic polynomial $y(t)=\rho_2+\rho_3t+\rho_4t^2$ such that $y(\phi(t))$ obtains the approximation order of 4 to $f(t)$, where $\phi'(t)\geq 0$, for all $t\in[0,1]$. Note that there are four unknowns ρ_i, $i=1,2,3,4$ in $y(\phi(t))$, we can introduce four constraints such that $y(\phi(t))$ interpolates either two points and their two derivatives (see also QT_1 in Section 2.2), or three points and one derivative of $f(t)$ (see also QT_2 in Section 2.3 and QT_3 in Section 2.4). From [6, Theorem 3.5.1, p. 67, Chapter 3.5], $y(\phi(t))$ can obtain both the approximation order of 4 and the convergence rate of 4, which is better than that of previous quadratic clipping method 3.

Generally speaking, QT_1 and QT_2 are used for convex cases that $(t,f(t))$ has no inflexion point, while QT_3 is applied for other cases that QT_1 and QT_2 fail. In principle, both QT_1 and QT_2 can construct two polynomials which can be used to bound $f(t)$, and the corresponding computation complexity is linear, which is much better than those of previous clipping methods.

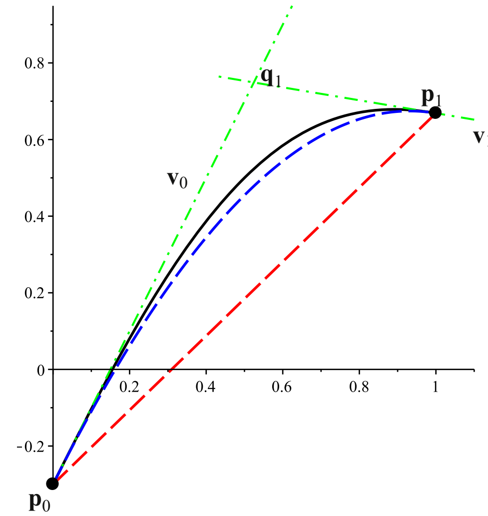

Fig. 1. Illustration of QT_1. The curve in solid black is the curve $\overline{\mathbf{C}}(t)$, while the curve in dashed blue is the quadratic Bézier curve with control points $\mathbf{p}_0$, $\mathbf{q}_1$ and $\mathbf{p}_1$. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this article.)

2.2. The first planar quadratic clipping method (QT_1)

Suppose that $\lambda_0\cdot\lambda_1<0$, e.g., when $\overline{\mathbf{C}}(t)$ is convex within $[0,1]$, see also Fig. 1. Let $\mathbf{q}_1$ be the intersection point between the two lines $\mathbf{p}_0+\mathbf{v}_0t$ and $\mathbf{p}_1+\mathbf{v}_1t$, which can be computed by solving the following constraints:

$$\mathbf{q}_1=\mathbf{p}_0+\alpha_0\mathbf{v}_0=\mathbf{p}_1-\alpha_1\mathbf{v}_1, \tag{2}$$

where α_0 and α_1 are two unknown real numbers. From Eq. (2), we have that

$$\mathbf{A}_1(i)=\mathbf{p}_i,\mathbf{A}_1'(i)=2\alpha_i\mathbf{v}_i,\quad i=0,1, \tag{3}$$

which means that $\mathbf{A}_1(u)$ interpolates two end-points and two directional tangent vectors of the two end-points. It can be verified that the values of α_0 and α_1 are positive. In fact, if the length of interval $[a,b]$ tends to be zero, then $\overline{x}_1(u)$ tends to be u, which means that the value of α_i tends to be 0.5, i=0, 1.

In this case, let $\phi_1(t)=t+\rho_{1,0}(t-1)t$, $\overline{y}_1(t)=p_0B_0^2(u)+\rho_{1,1}B_1^2(u)+p_1B_1^2(u)$ and $\overline{Y}_1(t)=\overline{y}_1(\phi_1(t))$, where $\rho_{1,i}$ is a unknown. The equation system

$$p_i=\overline{Y}_1(i),\quad d_i=\overline{Y}_1'(i),\quad i=0,1, \tag{4}$$

can be simplified into

$$\begin{cases} 2(p_1-p_0)\rho_{1,0}^2+(d_0-d_1)\rho_{1,0}+2p_0+d_1-2p_1+d_0=0, \\ \rho_{1,1}=p_1-\dfrac{d_1}{2(1+\rho_{1,0})}. \end{cases}$$

There may be two values of $\rho_{1,0}$, and we select the one which is of the smaller absolute value. Let $\overline{H}_1(t)=\overline{f}(t)-\overline{Y}_1(t)$. Combing Eq. (4) with [6, Theorem 3.5.1, p. 67, Chapter 3.5], there exists $\xi_1(t)\in[0,1]$ such that

$$\overline{f}(t)-\overline{Y}_1(t)=\frac{\overline{H}_1^{(4)}(\xi_1(t))}{24}(t-1)^2t^2. \tag{5}$$

2.3. The second planar quadratic clipping method (QT_2)

Let $\upsilon_0=\det(\mathbf{p}_m-\mathbf{p}_0,\mathbf{v}_m)$, $\upsilon_1=\det(\mathbf{p}_1-\mathbf{p}_m,\mathbf{v}_m)$ and $\upsilon_2=\det(\mathbf{p}_1-\mathbf{p}_0,\mathbf{v}_m)$. Suppose that $\upsilon_0\cdot\upsilon_1<0$, e.g., when $\overline{\mathbf{C}}(t)$ is convex within