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Dimension-independent multi-resolution Morse complexes ☆

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ABSTRACT

Morse and Morse–Smale complexes have been recognized as a suitable model for representing topological information extracted from discrete scalar fields. Here, we propose a dimension-independent multi-resolution model for Morse complexes built on a graph representation of the complexes, that we call a *Multi-Resolution Morse Incidence Graph* (*MMIG*). We define data structures for encoding the *MMIG* and we discuss how to extract from an *MMIG* topological representations of the scalar field over its domain M at both uniform and variable resolutions. We present experimental results evaluating the storage cost of the data structures encoding the *MMIG*, and timings for building and querying an *MMIG*.

1. Introduction

Morse theory offers a natural and intuitive way of analyzing the structure of a scalar field as well as of compactly representing it through a decomposition of its domain M into meaningful regions associated with the critical points of the field. These complexes are widely used in shape analysis and modeling, and they have been applied in scientific visualization for understanding and analyzing the critical features of a scalar field.

Simplification of Morse and Morse–Smale complexes has been an important research topic in these last years [1–3]. We have defined atomic simplification (and the inverse refinement) operators [3], which have the important property of forming a minimally complete basis for modifying Morse and Morse–Smale complexes in arbitrary dimension. Here, we define and implement such operators on a graph-based dual representation of Morse complexes called a *Morse incidence graph* (MIG) [4].

A multi-resolution representation of the topology of a scalar field f is crucial for interactive analysis and exploration of terrains, static and time-varying volume data sets, in order to maintain and analyze their features at different levels of detail and reduce the size of their representation. Here, we propose a dimension-independent graph-based model for multi-resolution representation of Morse and Morse–Smale complexes, which describes the topology of scalar fields in arbitrary dimensions, that we call a *Multi-Resolution Morse Incidence Graph* (*MMIG*), and we describe the data structures we have designed and implemented for encoding it.

An *MMIG* organizes several graph representations of domain M and scalar field f at different resolutions, and is capable of supporting the extraction of the graph which best approximates the topology of the field under given requirements depending on the specific application. It is possible to select from an *MMIG* a variety of graphs in which the resolution (defined by a suitable error criterion) is uniform or varies over the domain M of the scalar field. The multi-resolution model, its implementation described here, and the algorithms for querying it at uniform and variable resolutions will greatly enhance the analysis and understanding of static and dynamic volume data sets, which can be modeled as 3D and 4D scalar fields.

2. Background and related work

A C^2 real-valued function f over a closed compact n-dimensional manifold M is a *Morse function* if all its critical points are non-degenerate [1,5]. Integral lines that converge to a critical point p of index i form an i-cell called a *descending cell* of p. Dually, integral lines that originate at p form its ascending $(n-i)$-cell. The descending and ascending cells decompose M into *descending* and *ascending Morse complexes*, denoted as Γ_d and Γ_a, respectively (see Fig. 1(a) and (b) for a 2D example). A Morse function f is called a *Morse–Smale function* if and only if each non-empty intersection of a descending and an ascending cell is transversal. The connected components of the intersection define a *Morse–Smale complex*, which decomposes M into cells defined by integral lines with the same origin and destination.

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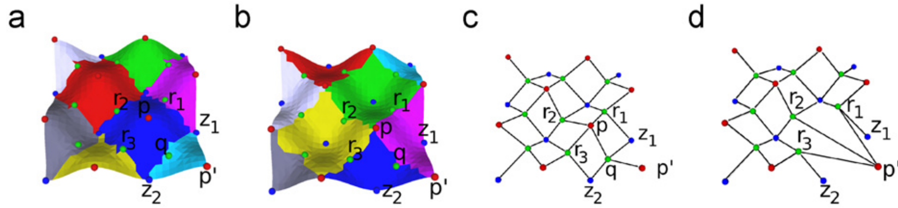

Fig. 1. The descending (a) and ascending (b) 2D Morse complex for function $f(x,y) = \cos x \cdot \cos y$, (c) the corresponding *MIG* and (d) *MIG* after application of $remove_{1,2}(q,p,p')$.

Algorithms for decomposing the domain M of f into an approximation of a Morse or of a Morse–Smale complex in 2D can be classified as *boundary-based* or *region-based*. In [6–8], algorithms for extracting the Morse–Smale complex from a tetrahedral mesh have been proposed. Discrete methods rooted in the discrete Morse theory proposed by Forman [9] are computationally more efficient [10,11]. For a survey, see [12,13].

Simplification of Morse and Morse–Smale complexes can be achieved by applying an operator called *cancellation* [1]. This operator has been investigated in 2D [14–17] and 3D [2] Morse–Smale complexes. In any dimension higher than two, a cancellation decreases by two the number of critical points (0-cells) in the Morse–Smale complex, but it may increase the number of higher-dimensional cells. Thus, a cancellation cannot be considered as a simplification operator, since it increases the size of the representation [18]. Several strategies have been proposed to postpone a cancellation that would introduce a number of arcs greater than a predefined threshold [18]. Atomic simplification operators have been defined in [3], which always reduce the number of cells in the Morse–Smale complex. These operators will be briefly reviewed in Section 3. Not much work has been done on modifying the scalar field f after a cancellation, thus coupling the topological simplification with the smoothing of f, and both are for 2D scalar fields [14,19].

3. Representation and simplification of Morse complexes

We briefly describe here two tools that we use in this work, namely a dimension-independent incidence-based graph representation for Morse complexes, that we call the *Morse Incidence Graph* (*MIG*) [4], and simplification and refinement operators on Morse complexes [3].

The *Morse Incidence Graph* (*MIG*) encodes the topology of both the ascending and descending Morse complexes Γ_a and Γ_d, respectively (see Fig. 1(c) for a 2D example). An *MIG* is a multigraph $G=(N,A)$ such that (i) there is a one-to-one correspondence between the nodes in N and the i-cells of Γ_d (and thus the $(n-i)$-cells of Γ_a), called *i-nodes* and (ii) there are k arcs joining an i-node p with an $(i+1)$-node q if and only if i-cell p appears k times on the boundary of $(i+1)$-cell q in Γ_d.

We use a dimension-independent data structure for representing the *MIG*, called the *incidence-based data structure* [4], by coupling the topology of the Morse complexes with the geometry of the underlying mesh Σ. The geometry is encoded only for the descending cells of the maxima and for the ascending cells of the minima. Such cells are encoded as a set of indexes of n-simplexes in the data structure representing Σ. These are the n-simplexes forming the descending cell of each maximum p in Γ_d and those forming the ascending cell of each minimum q in Γ_a. In [20], a dimension-specific data structure for 3D Morse–Smale complexes is proposed, in which the topological part is equivalent to the *MIG*. Such data structure stores the geometry of all cells in the Morse complexes and, thus, it requires more space than the 3D instance of the incidence-based representation.

The second tool that we use are the simplification and refinement operators on Morse complexes [3]. The *remove* simplification operator collapses two saddle points of consecutive index that are connected through a unique integral line. It has two instances, namely $remove_{i,i+1}$ and $remove_{i,i-1}$, for $1 \le i \le n-1$. There are two types of $remove_{i,i+1}$, denoted as $remove_{i,i+1}(q,p,p')$ and $remove_{i,i+1}(q,p,\emptyset)$, respectively.

Operator $remove_{i,i+1}(q,p,p')$ applies when i-saddle q is connected to $(i+1)$-saddle p and exactly one other $(i+1)$-saddle p' different from p. It collapses i-saddle q and $(i+1)$-saddle p into $(i+1)$-saddle p'. In the descending complex Γ_d, it collapses i-cell q and $(i+1)$-cell p into a unique $(i+1)$-cell p' incident in i-cell q and different from $(i+1)$-cell p.

Operator $remove_{i,i+1}(q,p,\emptyset)$ deals with the situation in which i-saddle q is connected to only one $(i+1)$-saddle p. It eliminates i-saddle q and $(i+1)$-saddle p from the set of critical points of scalar field f, i.e., it eliminates i-cell q and $(i+1)$-cell p from Γ_d. The $remove_{i,i-1}(q,p,p')$ operator is completely dual. For brevity, we will consider only operators of the first type.

The refinement *insert* operator, inverse to *remove* operator, is defined as an undo of a *remove*. It has two instances, namely $insert_{i,i+1}$ and $insert_{i,i-1}$ inverse to $remove_{i,i+1}$ and $remove_{i,i-1}$, respectively.

It has been shown in [3] that the *remove* and *insert* operators form a minimally complete basis for the set of topologically consistent operators for updating Morse complexes on a manifold M in any dimension. In particular, the general cancellation operator defined in Morse theory [1], which cancels two critical points of consecutive index connected through a unique integral line, can be expressed as a suitable combination of *remove* and *insert* operators.

4. The multi-resolution Morse incidence graph (*MMIG*)

In this section, we define a multi-resolution model for the topology of Morse and Morse–Smale complexes, that we call a *Multi-Resolution Morse Incidence Graph* (*MMIG*). An *MMIG* is generated from the *MIG* representing the two ascending and descending Morse complexes at full resolution by iteratively applying *remove* operators in increasing order of persistence. Intuitively, *persistence* measures the importance of the pair of critical points p and q to be eliminated, and is equal to the absolute difference in function values between them.

A *remove* operator transforms an *MIG* $G=(N,A)$ into a simplified *MIG* $G' = (N',A')$ by eliminating the two nodes corresponding to the two critical points eliminated from N, and by suitably reconnecting the remaining nodes in G'. A $remove_{i,i+1}(q,p,p')$ is feasible on G if there exists a unique arc (p,q) in A, and there exists a unique $(i+1)$-node $p' \in N$ different from $(i+1)$-node p and connected to i-node q. In order to define the effect of $remove_{i,i+1}$, we consider three sets of nodes in G, namely the set $Z=\{z_h, h=1,\dots,h_{max}\}$ of the $(i-1)$-nodes connected to i-node q; the set $S=\{s_k, k=1,\dots,k_{max}\}$ of the $(i+2)$-nodes connected to