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Technical Section

Quadratic curve and surface fitting via squared distance minimization

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ABSTRACT

Quadratic curve and surface fitting to a set of data points are fundamental problems in reverse engineering and many other application areas. We develop the fitting methods for quadratic curves and surfaces based on the squared distance minimization technology. The basic idea of squared distance minimization for curve and surface fitting is first presented. Then we devise the corresponding squared distance term for each quadratic curve and surface, and minimize it to obtain its parameters. We repeat the squared distance minimization and update the parameters of the quadratic curve and surface by iterations until convergency. Consequently, the final fitting result is achieved. Experimental results demonstrate the effectiveness of the fitting method.

1. Introduction

Quadratic curves and surfaces are used widely for many applications, including CAD and geometric modeling, metrology, computer graphics, computer vision, pattern recognition, and so forth. Particularly, in reverse engineering, there is a general demand to reconstruct the 3D model of the geometric shape from the scanning data points, i.e. to construct a boundary representation solid model of the object's shape. Quadratic curves and surfaces are frequently existing in the composition of mechanical products. Therefore, the considerable attention has been attracted to the approximation of a set of data points by quadratic curves and surfaces [1–8]. Basically, curve and surface fitting can be translated into solving an optimization problem, i.e., minimizing the objective function of a fitting error term. Therefore, how to define the fitting error term is key to curve and surface fitting. A number of researchers have been working on this topic in two decades, and hence many definitions regarding the fitting error term are proposed. Basically, those definitions mainly consist of four types: (1) algebraic distance error term (AD) [9], (2) Euclidean distance error term (ED) [10,11], (3) tangent distance error term (TD) [12] and (4) squared distance error term (SD) [13–15].

Here we introduce those four typical error terms briefly. Let $P=\{p_1,p_2,\ldots,p_m\}$ be scanning data points from a target shape, it can be approximated by $S(\mathbf{x})$, where $\mathbf{x}$ is the state vector of the closest point of $p\in P$ on S. Suppose the corresponding implicit expression of S is $F(p)=0$, then we have the algebraic distance error $Error_{AD}$, the Euclidean distance error $Error_{ED}$, the tangent distance error $Error_{TD}$ and the square distance error as follows:

$$Error_{AD}=\sum_{i=1}^{m}F^2(p_i) \tag{1}$$

$$Error_{ED}=\sum_{i=1}^{m}\|S(\mathbf{x}_i)-p_i\|^2 \tag{2}$$

$$Error_{TD}=\sum_{i=1}^{m}\|[S(\mathbf{x}_i)-p_i]\cdot N_i\|^2 \tag{3}$$

$$Error_{SD}=\sum_{i=1}^{m}\left(\sum_{j=1}^{3}\alpha_{i,j}[(S(\mathbf{x}_i)-p_i)\cdot N_{i,j}]^2\right) \tag{4}$$

where N_i is the unit normal vector of $S(\mathbf{x}_i)$ in Eq. (3); $N_{i,j}$ is the unit vector of the y-axis in the local Cartesian frame of $S(\mathbf{x}_i)$ on S and $\alpha_{i,j}$ is the non-negative coefficient in Eq. (4), which will be explained in detail in the next section.

The algebraic distance is incapable of measuring the accurate distance between a fitting object and the target shape, which is generally used for initial object fitting. The ED-based fitting method is widely used in computer graphics and CAGD applications thanks to its simplicity and straightforwardness. However, the convergence of the ED-based method is slow relatively and it is often trapped in a poor local minimum. The TD-based fitting method is quite popular in the computer vision community. It converges faster than the ED-based fitting method. Nevertheless, it is not stable near a high-curvature part of a target shape.

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By contrast, the SD error term faithfully measures the geometric distance between a fitting object and a target shape, thus leading to faster and more stable convergence than the ED error term, and the TD error term [14].

Accordingly, we develop the fitting methods for the most common curves and surfaces in mechanical products, i.e. quadratic curves and surfaces, using the SD error term. Specifically, the fundamentals of squared distance minimization are presented firstly. Then, the squared distance error terms are designed for the quadratic curves, including circle, ellipse, parabola and hyperbola, as well as the quadratic surfaces, such as cylinder, sphere and cone, respectively. Furthermore, based on those error terms, the method using the squared distance minimization strategy is given in details to extract the corresponding parameters of each quadratic primitive.

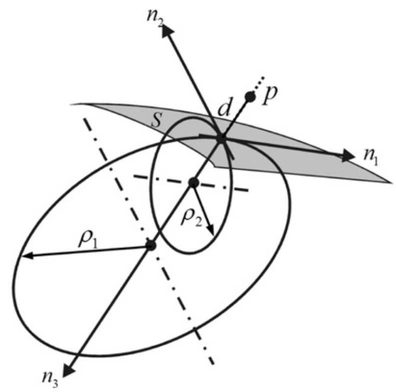

Fig. 2. Frenet frame of 3D surface.

2. Fundamentals of squared distance minimization

We first introduce the squared distance of 2D curve briefly. For more details and discussions, we refer to [13–15]. Given a point set $P=\{p_1,p_2,\ldots,p_m\}\subset R^2$ in a plane Π, the fitting curve is represented with $c(t)$ in Fig. 1. Let $c(t_0)$ be the closest point of a point $p\in P$ on $c(t)$, d the shortest distance, and the curvature, curvature radius and curvature center of $c(t_0)$ on $c(t)$ are k, $\rho=1/|k|$, and $e(t_0)$, respectively. According to the curve Theory [16], the Frenet frame of a point with the parameter of t on $c(t)$ can be represented by the tangent vector $\alpha(t)$ and the normal vector $\beta(t)$ of $c(t)$. Consequently, a local Cartesian coordinate system xoy on $c(t_0)$ in Π could be constructed, where the origin, x-, y-axis are $c(t_0)$, $\alpha(t_0)$, $\beta(t_0)$, respectively. Under xoy, the local coordinates of p and $e(t_0)$ are $(0,d)$ and $(0,-\rho)$.

Let $q=(x,y)$ be in a small neighborhood of p, the squared distance D^2 from q to $c(t)$ can be approximately expressed with

$$D^2(x,y)=(\|q-e(t_0)\|-\rho)^2=\left(\sqrt{x^2+(y-\rho)^2}-\rho\right)^2 \tag{5}$$

The second-order Taylor approximant f of the function D^2 at q is

$$f(x,y)=\frac{d}{d-\rho}x^2+y^2 \tag{6}$$

By transforming it into the global coordinate system, it becomes

$$F(x,y)=\frac{d}{d-\rho}[\alpha(t_0)\cdot(q-c(t_0))]^2+[\beta(t_0)\cdot(q-c(t_0))]^2 \tag{7}$$

which, strictly speaking, is the second order approximation of the squared distance function of 2D curve. Since it is derived from a direct attempt to accurately approximate the squared distance function, we refer to it as the squared distance function for simplicity. Note that $F(x,y)$ may have a negative value if $0<d<\rho$, which may lead to the failure of the following optimization iterations. In this case, we remove all negative items in the squared distance function as [14].

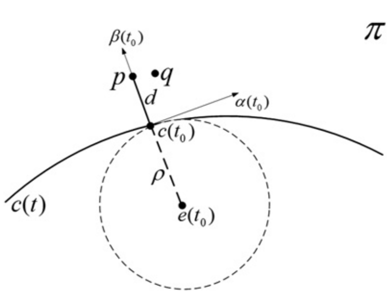

Fig. 1. Frenet frame of 2D curve.

Then, we discuss the 3D surface case. Let S be the fitting surface of a point set $P=\{p_1,p_2,\ldots,p_m\}\subset R^3$, s is the closest point of a point $p\in P$ on S and d is the shortest distance. The principal curvatures of s on S are k_1, k_2, and the corresponding curvature radii are $\rho_1=1/|k_1|$ and $\rho_2=1/|k_2|$, see Fig. 2. According to the surface Theory [16], the Frenet frame of s on S consists of the two vectors n_1, n_2 of the principal curvature directions, and the normal vector n_3. Applying the squared distance function of 2D curve here, the squared distance function of 3D surface can be expressed

$$F(p)=\frac{d}{d-\rho_1}[n_1\cdot(p-s)]^2+\frac{d}{d-\rho_2}[n_2\cdot(p-s)]^2+[n_3\cdot(p-s)]^2 \tag{8}$$

Therefore, the squared distance function of each point $p_i\in P$ to S is

$$\begin{aligned}F(p_i)&=\frac{d_i}{d_i-\rho_{i,1}}[n_{i,1}\cdot(p_i-s_i)]^2+\frac{d_i}{d_i-\rho_{i,2}}[n_{i,2}\cdot(p_i-s_i)]^2+[n_{i,3}\cdot(p_i-s_i)]^2\\&=\sum_{j=1}^{3}(\alpha_{i,j}\cdot[n_{i,j}\cdot(p_i-s_i)]^2)\end{aligned} \tag{9}$$

where $\alpha_{i,1}=d_i/(d_i-\rho_{i,1})$, $\alpha_{i,2}=d_i/(d_i-\rho_{i,2})$ and $\alpha_{i,3}=1$.

Let $\mathcal{P}$ be the parameter vector of surface S, S^+ denotes the fitting surface with the updated parameter vector $\mathcal{P}^+=\mathcal{P}+\mathcal{D}$, where $\mathcal{D}$ is incremental updates to S. The closest point of p_i on S^+ is s_i^+, which is different from the shortest distance s_i of p_i to S. However, because the difference is quite small, we can substitute s_i^+ with s_i approximately. Suppose that $\mathbf{x}_i$ is the state vector of s_i, Eq. (8) can be transformed to

$$E(\mathcal{P}^+)=\sum_{j=1}^{3}(\alpha_{i,j}\cdot[n_{i,j}\cdot(p_i-S^+(\mathbf{x}_i))]^2) \tag{10}$$

For all points of P, the squared distance function is

$$F(\mathcal{P}^+)=\sum_{i=1}^{m}E(\mathcal{P}^+)=\sum_{i=1}^{m}\sum_{j=1}^{3}(\alpha_{i,j}\cdot[n_{i,j}\cdot(p_i-S^+(\mathbf{x}_i))]^2) \tag{11}$$

By minimizing this squared distance function, i.e.

$$\min F(\mathcal{P}^+)=\min\sum_{i=1}^{m}\sum_{j=1}^{3}(\alpha_{i,j}\cdot[n_{i,j}\cdot(p_i-S^+(\mathbf{x}_i))]^2) \tag{12}$$

the updated parameter vector $\mathcal{P}^+$ of S^+ is obtained.

3. Quadratic curve and surface fitting

Based on this squared distance minimization method, we propose the fitting methods of quadratic curves and surfaces,