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# An alternative formulation for the fast multipole method

Giuseppe Borzì *

*Dipartimento di Ingegneria Civile, Informatica, Edile, Ambientale e Matematica Applicata, Università di Messina Panoramica dello Stretto, Messina I-98166, Italy*

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ABSTRACT

This paper describes an alternative formulation for the fast multipole method based on spherical waves decomposition. It is somewhat simpler to implement than the standard fast multipole method and also better suited at low frequencies. The new formulation is mainly based on a technique for the interpolation of the bistatic radar cross section derived from the Wacker's method for antenna measurements.



## 1. Introduction

The fast multipole method (FMM) is widely used to reduce the complexity of problems that requires the discretization of an integral equation. It was first proposed by Rokhlin [1] to deal with particle simulations and for the fast solution of static integral equations and subsequently extended to solve scalar and vector scattering problems. It is a subject currently included in advanced computational electromagnetics books [2,3].

The basic, two level FMM, reduces the memory and computational complexity of the discrete integral equation from $O(N^2)$ to $O(N^{1.5})$ [4], and its multi level version, known as the multilevel fast multipole algorithm (MLFMA), further reduces the complexity to $O(N\log(N))$ [2].

The FMM is based on the decomposition of the domain into cubic subdomains where the local field is approximated by a superposition of plane waves, which are translated to far cubes using appropriate operators.

A well-known problem of the FMM is the low frequency breakdown which is due to the loss of accuracy caused by the computation of spherical Bessel functions of the second kind at very low arguments [2]. Some modification of the FMM, such as the low frequency MLFMA (LF-MLFMA) [2], the stable plane wave expansion (SPW-FMM) [5] and the accelerated Cartesian expansion (ACE) [6] have been successfully developed to solve this problem.

In this paper an alternative formulation for the two-level FMM is introduced. Like the standard FMM the domain is divided into non-overlapping cubes and near cubes interaction is treated by direct integration of the Green's equation. But the local fields interaction between far cubes is dealt with using a decomposition into spherical waves, and the corresponding translations.

The Wacker method [7] is used to determine the coefficients of the outgoing spherical waves for each cube. This method is used in antenna measurement, and has been adapted to numerical problems [8], namely for the interpolation of bistatic radar cross section (BRCS).

The next step determines the translations, i.e. the expansion of outgoing spherical waves from a cube as incident spherical waves on non-nearby cubes. This is accomplished using a modified Wacker method. Finally, the interaction of the incident spherical waves with the scatterer inside each cube is computed.

Currently, the spherical wave decomposition based FMM (SWD-FMM) is not competitive with the standard FMM, given the amount of research, and hence refinement, the latter has enjoyed. Nevertheless, it can become an interesting alternative which, to the author's opinion, is worth exploring. It is relatively easy to code, the most difficult part being the implementation of the Wacker's method, and is less susceptible to the low frequency breakdown than the standard FMM. Although it is not as good at low frequencies as the above mentioned modifications.

The paper is structured as follows: Wacker's method is briefly described in Section 2, then Section 3 gives a detailed presentation of the alternative method. In Section 4 the method is applied to some scattering problems. The conclusion follows in Section 5.

The engineer's convention for time dependence $e^{j\omega t}$ is used throughout the paper, $\lambda$ denotes the wavelength and $k$ the wavenumber.

* Tel.: +39 0903977323; fax: +39 0903977480.
*E-mail addresses:* gborzi@ieee.org, gborzi@unime.it

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## 2. A brief description of the Wacker method

The Wacker method interpolates the BRCS by first computing the scattering coefficients $\alpha_{slm}^{(out)}$ of the spherical waves expansion of the outgoing field $\bar{E}^{(out)}$ outside a sphere enclosing the scatterer

$$\bar{E}^{(out)} = \sum_{s=1}^{2}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\alpha_{slm}^{(out)}\bar{F}_{slm}^{(out)} \tag{1}$$

where the summation on $l$ can be truncated because the scattering coefficients vanish as $l\to\infty$. For scattering problems the maximum value for $l$ can be estimated from the scatterer's size [9]. In the subsequent formulas the truncated series up to order $l_x$ will be denoted by $\sum_{l=1}^{l_x}$ omitting the implied $s$ and $m$ summations.

The tangential components of $\bar{F}_{slm}^{(out)}$ on a spherical surface are given by

$$z_{sl}^{(out)}\bar{X}_{slm} \tag{2}$$

where $\bar{X}_{slm}$ is the vector spherical harmonics (VSH) of indexes $s$, $l$, $m$ and

$$\begin{aligned} z_{1l}^{(out)}(kr) &= h_l^{(2)}(kr) \\ z_{2l}^{(out)}(kr) &= \frac{1}{kr}\frac{d}{dr}rh_l^{(2)}(kr) \end{aligned} \tag{3}$$

with $h_l^{(2)} = j_l - j\,y_l$ being the spherical Hankel function of the second kind. Using the orthonormality property of the VSH, the scattering coefficients can be computed as

$$\alpha_{slm}^{(out)} = \frac{1}{z_{sl}^{(out)}(kr)}\oint_{S_r}\bar{E}^{(out)}\cdot[\bar{X}_{slm}(\theta,\phi)]^{*}ds \tag{4}$$

where $S_r$ is a spherical surface of radius $r$ enclosing the scatterer and the asterisk denotes complex conjugation.

If the scattered field $\bar{E}^{(out)}$ has negligible spherical waves coefficients for $l > l_x$, its $\widehat{\theta}$ and $\widehat{\phi}$ components are given by finite linear combinations of complex exponentials in $\theta$ and $\phi$ up to order $l_x$. Also, $\bar{X}_{slm}(\theta,\phi)\sin\theta$ is a finite linear combinations of complex exponentials in $\theta$ up to order $l+1$ multiplied by $e^{jm\phi}$.

Once the coefficients of these linear combinations are determined, formula (4) reduces to a linear combination of integrals of the form $\int_0^{\pi}d\theta\int_0^{2\pi}d\phi e^{j(n\theta+m\phi)}$.

The coefficients of the trigonometric interpolation are easily calculated by means of a two-dimensional fast Fourier transform (FFT) [10] for $\bar{E}^{(out)}$ and a one-dimensional FFT for the VSH. However, before applying the FFT to the $\widehat{\theta}$ and $\widehat{\phi}$ components of $\bar{E}_s(\bar{r})$ and $\bar{X}_{slm}(\theta,\phi)\sin\theta$, they must be extended to $\theta > \pi$. Using the definition of the VSH it is easy to see that

$$\begin{aligned} \widehat{\theta}\cdot\bar{X}_{slm}(\theta,\phi) &= -\widehat{\theta}\cdot\bar{X}_{slm}(2\pi-\theta,\phi\pm\pi) \\ \widehat{\phi}\cdot\bar{X}_{slm}(\theta,\phi) &= -\widehat{\phi}\cdot\bar{X}_{slm}(2\pi-\theta,\phi\pm\pi) \end{aligned} \tag{5}$$

and the same relationships also hold for $\bar{E}^{(out)}$.

Assuming that the series (1) is truncated at $l_x$, the number of equally spaced sampling points along each angular spherical coordinate needed by the FFT, is $2l_x+1$, so that their total number is $2l_x(2l_x+1)+1$. This number can be nearly halved if the number of sampling points along $\phi$ is even. In this case for each sampling point $P$ having $\theta_P > \pi$ there is a sampling point $Q$ with $\theta_Q = 2\pi - \theta_P$, $\phi_Q = \phi_P \pm \pi$, so that the samples in $P$ can be derived through (5) from the samples in $Q$. In this way, the total number of sampling points reduces to $2(l_x+1)l_x+1$.

The number of sampling points along the extended $\theta$ can be made equal to the number of sampling points along $\phi$ by adding a single sampling point at $\theta=\pi$, so that the number of sampling points becomes $2(l_x+1)l_x+2$. The set of equally spaced sampling points and their associated $\widehat{\theta}$ and $\widehat{\phi}$ versors on a spherical surface of radius $r$ with center $c$ for order $l$ will be denoted by $S(l, r, c)$.

When $l_x$ is not known beforehand, one can start with an estimated value and check that there is an integer $l_e < l_x$ such that

$$\sum_{l=l_e+1}^{l_x}\left|\alpha_{slm}^{(out)}\right|^2 < \epsilon\sum_{l=1}^{l_x}\left|\alpha_{slm}^{(out)}\right|^2 \tag{6}$$

where $\epsilon$ is a user specified tolerance. If $l_e = l_x$, then $l_x$ is increased and the process repeated until $l_e < l_x$. Note that $\left|\alpha_{slm}^{(out)}\right|^2$ is proportional to the outgoing power of the corresponding wave, hence (6) can be interpreted as a power-based truncation, i.e. the waves whose outgoing power is small are discarded.

## 3. The alternative formulation

As mentioned in Section 1, the integration domain is divided into non-overlapping cubes and each element is assigned to a cube if its center lies in the cube. This means that there are variables belonging to more than one cube and that elements assigned to a cube may not be entirely contained in the cube. A more memory efficient implementation would assign each variable to just one cube, but it is harder to implement.

Near cubes interaction is computed by direct integration of the Green's equation, and for each cube a matrix that relates the variables of the cube with its scattering coefficients is computed. These matrices will be referred to as scattering matrices. Next, for each translation that relates a cube couple $(a, b)$, a matrix that gives the coefficients of the expansion into incident spherical waves centered in $b$ of the outgoing spherical waves centered in $a$ is calculated. Obviously, such matrix merely depends on the difference $p = a - b$.

The translation matrices have some symmetries that allow a significant reduction in computing time and memory usage. The last step is the computation of the boundary conditions due to the incident spherical waves centered at each cube.

### 3.1. Scattering matrices

The scattering matrix for a cube gives the scattering coefficients of the cube's field as a linear function of the variables $V_c$ of the elements belonging to the cube

$$\bar{\alpha}^{(out)} = \begin{bmatrix}\vdots \\ \alpha_{slm}^{(out)} \\ \vdots\end{bmatrix} = S_{(l_s,c)}V_c \tag{7}$$

where $c$ is the cube's center and $l_s$ is the maximum order of the outgoing spherical waves. Its construction requires the selection of appropriate values for $r$ and $l_s$ with which to build the set $S(l_s, r, c)$ used by Wacker's integration.

For the radius the value is assigned as $r = \max(10*\delta, 2\lambda)$ where $\delta$ is the cube size. In exact arithmetic the final result would not depend on the value of $r$ as long as the corresponding sphere encloses the cube, but with computer arithmetic the radius must be much greater than the cube size.