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# Fractional Euler analog-to-digital transform

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ABSTRACT

In this paper, the design of fractional Euler transform which is the regular Euler transform where the digital delay operator $z^{-1}$ is replaced by the ideal digital fractional delay operator $z^{-\alpha}$ is presented. This introduced transform will be used in analog-to-digital transform to improve the accuracy of a digital filter equivalent to a given analog filter. However, because of the fractional delay operator $z^{-\alpha}$ the proposed fractional Euler transform cannot be exactly implemented; only a limited implementation by means of infinite impulse response (IIR) digital filter can be achieved using approximation techniques. First, the fractional order $\alpha$ of the digital fractional delay operator $z^{-\alpha}$, for $0<\alpha<0.5$, is selected such that the relative error between the analog differentiator $s$ and its equivalent digital one obtained by the fractional Euler transform is minimum. Then, the ideal digital fractional delay operator $z^{-\alpha}$ is approximated by a digital IIR filter based on approximation of analog fractional order system leading to an IIR digital filter implementation of the fractional Euler transform. Illustrative examples are given to show the effectiveness of the fractional Euler analog-to-digital transform design. Finally, the design of low order digital differentiator using the proposed fractional Euler transform has been compared to the most recent designed analog-to-digital transforms.



## 1. Introduction

The digital differentiator is a very useful tool in digital signal processing. When designing a digital filter equivalent to an analog one, impulse invariance, step invariance, Euler transform and bilinear transform are used to perform the analog-to-digital transform (ADT) [1]. Although these methods are simple and easy to use to fit the ideal differentiator, they have large magnitude distortion in the high-frequency range. To enhance the fitting of the ideal differentiator in high-frequencies, several design techniques have been proposed such as Al-Alaoui transform [2], Taylor series method [3], the quadratic programming method [4], wideband differentiator [5], differentiator using fractional delay filter [6,7] and fractional bilinear transform [8]. More recently, wideband recursive digital differentiators with relatively lower errors have been proposed in Upadhyay [9] and Al-Aloui [10].

Digital fractional delay filter has been used in several digital signal processing applications such as beam steering, time adjustment in digital receivers, time delay estimation, analog-to-digital transform [6–8,11–15]. The ideal transfer function of the digital fractional delay is given by:

$$D(z) = z^{-a}, \quad 0 < \alpha < 1. \tag{1}$$

Because it cannot be exactly implemented only a limited implementation can be achieved. Many design techniques have been proposed for digital finite impulse response (FIR) and IIR filter type to approximate the ideal digital fractional delay transfer function of (1) [15–22]. Besides, fractional order system theory is a very growing subject by its applications in different areas of physics and engineering [23–27].

In this paper, the fractional Euler transform which is the regular Euler transform where the digital delay operator $z^{-1}$ is replaced by the digital delay operator $z^{-\alpha}$, for $0<\alpha<0.5$, is proposed in ADT to improve the design accuracy at high frequency region compared to the existing ADT techniques. The ideal transfer function of the proposed fractional Euler transform is given by:

$$F(z) = \frac{(1 - z^{-\alpha})}{\alpha T}, \tag{2}$$

where $T$ is the sampling period and $0<\alpha<0.5$. First, the parameter $\alpha$ of the ideal digital fractional delay operator $z^{-\alpha}$, for $0<\alpha<0.5$, is selected such that the relative error between the analog differentiator s and its equivalent digital one obtained by the fractional Euler transform is minimum. Then, the chosen digital fractional

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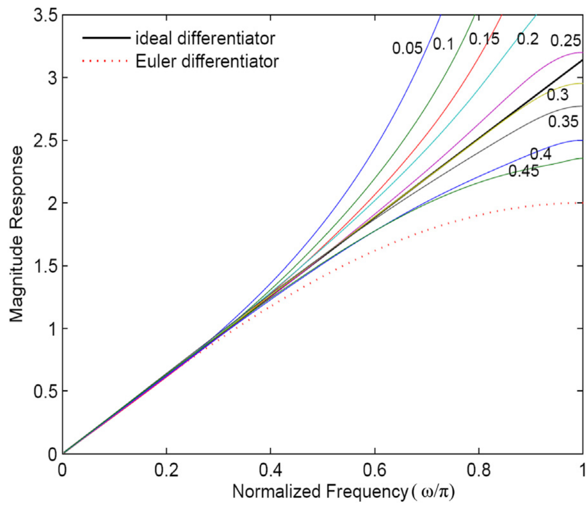


**Fig. 1.** Magnitudes of the Bode plot of the ideal analog differentiator $s=j\omega$ (dashed line), its digital equivalent using the Euler transform $(1-e^{-j\omega})/T$ (dotted line) and using the fractional Euler transform $F(e^{j\omega\alpha})$ for $T=1s$ and different values of $\alpha$.

delay operator $z^{-\alpha}$, for $0<\alpha<0.5$, is approximated by a digital IIR filter based on the approximation of analog fractional order system such that the proposed fractional Euler transform can be easily implemented. Finally, some design examples are illustrated to show the effectiveness of the proposed fractional Euler ADT design technique. In addition, a low order digital differentiator designed from fractional Euler ADT is compared to the most recent digital differentiators.

## 2. Analog-to-digital transform using fractional Euler transform

### 2.1. Motivation

Substituting $z=e^{j\omega}=e^{j\Omega T}$ such that $\omega=\Omega T$ in Eq. (2), we will get:

$$F(e^{j\omega})=\frac{(1-e^{-j\Omega T\alpha})}{\alpha T}. \tag{3}$$

The Taylor series expansion of the exponential function $e^{-j\Omega T\alpha}$ is given as:

$$e^{-j\Omega T\alpha}=1-j\Omega T\alpha+\sum_{k=2}^{\infty}\frac{(-j\Omega T)^k}{k!}\alpha^k. \tag{4}$$

When $\alpha$ approaches zero the above equation will be:

$$e^{-j\Omega T\alpha}\cong 1-j\Omega T\alpha. \tag{5}$$

Then (3) yields the following expression:

$$F(e^{j\omega})\cong\frac{\left[1-(1-j\Omega T\alpha)\right]}{\alpha T}\cong j\Omega. \tag{6}$$

Therefore, the digital filter $F(z=e^{j\omega})$ is equivalent to the analog differentiator $s=j\Omega$. Hence, the use of the Fractional Euler Transform $F(z)=(1-z^{-\alpha})/\alpha T$ to perform the ADT may reduce its error at high frequency range compared to the conventional transforms such Euler, Bilinear and Al-Alaoui transforms and the most recent ADT transforms presented in the literature.

### 2.2. Fractional Euler transform design

Fig. 1 shows the magnitudes of the Bode plot of the ideal analog differentiator $s=j\Omega$ and $F(z=e^{j\omega})$ of (3), for $T=1s$ and different values of $\alpha$. We note that the fractional Euler transform $F(e^{j\omega})$ has been multiplied by a factor $F_0$ because it has a little bias from the

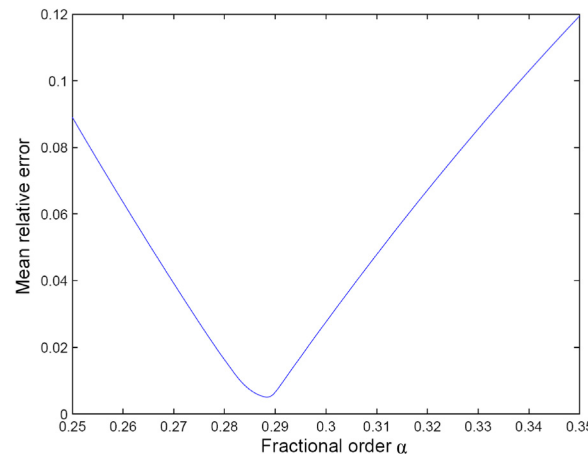


**Fig. 2.** Mean relative error versus the parameter $\alpha$.

theoretical value. It is also worth noting that for $T=1s$ the analog frequency $\Omega$ is replaced by the digital frequency $\omega$.

In Fig. 1 the digital differentiator $F(e^{j\omega})=(1-e^{-j\omega\alpha})/\alpha T$ equivalent to the analog one $s=j\omega$ is better than the Euler digital differentiator $(1-e^{-j\omega})/T$ for $0.25<\alpha<0.35$. So, to get the best parameter $\alpha$ such that $F(e^{j\omega})\cong j\omega$, we will use the mean relative error $Er_m(\alpha)$ given as:

$$Er_m(\alpha)=\frac{1}{Np}\sum_{i=1}^{Np}\left|\frac{\left|j\omega_i\right|-\left|(1-e^{-j\alpha\ \omega_i T})/\alpha T\right|}{\left|j\omega_i\right|}\right|, \tag{7}$$

where $Np$ is the number of points $\omega_i$ in the frequency band $[0,\pi]$ and $T=1s$. Fig. 2 shows the function $Er_m(\alpha)$ versus $\alpha$. In this case, $\alpha$ is varied from 0.25 to 0.35 with a step of 0.001.

From Fig. 2 $Er_m(\alpha)$ has a minimum value of 0.005 occurring at $\alpha_{min}=0.2885$. The factor $F_0$ of the bias used for the above range of the parameter $\alpha$ is $F_0=0.86313$. Hence, the proposed fractional Euler transform $F(z)$ which almost fits the ideal analog differentiator $s$ is given by:

$$F(z)=0.86313\,\frac{\left(1-z^{-0.2885}\right)}{0.2885}=3\left(1-z^{-0.2885}\right). \tag{8}$$

### 2.3. Digital IIR filter approximation of the ideal fractional delay

In this section, the method developed in [22] was used to design an IIR filter to approximate a given fractional delay $z^{-\alpha}$, for $0<\alpha<0.5$. The coefficients of the closed form digital IIR filter are based on the approximation of fractional order systems. First we have used Charef's approximation method to derive the analog rational function approximation, for a given frequency band, of the fractional power pole (FPP) [28]; then the Tustin generating function is used to digitize the FPP to obtain a closed form IIR digital filter which approximates the digital fractional delay operator $z^{-\alpha}$, for $0<\alpha<0.5$. Hence, we will get:

$$D(z)=z^{-\alpha}=(z+1)\,\frac{\prod_{i=0}^{N-1}\left(1+(2/(Tz_0(ab)^i))\right)(z-\bar{z}_i)}{\prod_{i=0}^{N}\left(1+(2/(Tp_0(ab)^i))\right)(z-\bar{p}_i)}$$

$$=\sum_{i=0}^{N}\frac{\bar{k}_i(1+z^{-1})}{(1-\bar{p}_i z^{-1})},\quad \text{for}\quad 0<\alpha<1, \tag{9}$$