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# Probability density of the phase of a random RF pulse in the presence of Gaussian noise☆

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**Abstract**

In this paper, we show that the probability density function (pdf) of the phase of a random nonstationary radio frequency (RF) impulse signal perturbed by Gaussian noise does not depend on the multiple time derivatives of the signal amplitude and phase at a given time instance and is the Bennett's pdf. Employing this revealing, we derive an alternative form of the conditional pdf of the phase representing it with the von Mises/Tikhonov pdf conditional on the envelope, signal-to-noise ratio (SNR), and signal phase. We expend this new form to the Fourier series and investigate for the locked and unlocked reference phases. The error probability for the phase to exceed a threshold is also analyzed.


***Keywords:*** Random RF pulse; Phase; Phase difference; Probability density; Gaussian noise

## 1. Introduction

In coherent pulse phase measurement systems [1], radars [2], remote passive wireless surface acoustic wave (SAW) sensing [3], etc., the phase is estimated employing the maximum likelihood function approach via the in-phase and quadrature-phase components formed for the reference phase. If an ideal receiver is assumed, the informative phase distribution is equal to that of the coherent estimate. Therefore, errors in such systems are associated with the probability properties of the phase of the received radio frequency (RF) pulse. The phase bears information about the time delay, Doppler, Doppler rate, etc., and is sensitive to the reference phase that may contribute to with regular and random errors. Moreover, the waveform of a RF pulse may suffer from fading. Accordingly, at some time instant $t_0$, both the waveform and phase may demonstrate multiple nonzero time derivatives at the receiver detector. This makes the joint probability density function (pdf) of the received RF pulse conditional and multivariate in the presence of Gaussian noise induced by the receiver.

Distribution of the phase of a signal perturbed by Gaussian noise is a fairly old problem having been under consideration for decades. In [4], Rice presented and studied the problem that was also investigated by Bunimovich [5], Norton et al. [6], and some other authors. In 1956, Bennett [7] derived the relevant pdf that is now used widely in a broad area of applications. The function appears from the joint pdf of the envelope and phase given by Rice in [8], by integrating over the envelope values from 0 to $\infty$ [9]. Because Bennett's pdf utilizes the probability integral, Matthews proposed in [10] to expand it to the Fourier series. This expansion also appears from the pdf of the phase difference between two uncorrelated vectors with one deterministic phase derived by Tsvetnov in 1969 [11]. Soon after, the phase pdf was studied by Weinstein in the approximate form in [12] and in the exact form in [13]. In 1988, Leib and

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Pasupathy proved that, by large signal-to-noise ratio (SNR), Bennett's pdf becomes Tikhonov's [14] that was also known from Tikhonov's works [15,16]. In those works, Tikhonov mentioned that Bennett's pdf is conditional on the vector phase and, in [17], Blackman exploited this fact relating the studies to the error probability in communication channels with differential phase-shift-keying (DPSK). Because fading in digital channels may cause errors, investigations have also been provided by many authors for the amplitude having Rayleigh, Rice, Nakagami, and, albeit not commonly, other distributions. For digital communication channels, the problems with fading were generalized in [18].

It has to be remarked now that the above mentioned and many other works consider an instantaneous signal as a constant, although randomly valued, existing in the presence of noise. Such an approximation is sufficient for the elementary rectangular bit. On the other hand, it remains unclear what happens if the RF pulse waveform and/or the phase function have nonzero time derivatives (e.g., in Gaussian waveform) or the derivative even does not exist at some point (e.g., in triangular waveform). Although, it is known that Bennett's pdf is still valid if the amplitude and phase of a signal change linearly (having nonzero first time derivatives[1]) at $t_0$ [16,20].

In this paper, we prove that the pdf of the phase of a random nonstationary RF pulse shaped with any waveform does not depend in the presence of Gaussian noise on the time derivatives of the amplitude and phase at $t_0$. Bennett's conditional pdf is thus exhaustive. The rest of the paper is organized as follows. In Section 2, we discuss the signal model and formulate the problem. Section 3 begins with the derivation of the six-variable conditional pdf of the signal envelope and phase and their first and second time derivatives. It then proceeds with the derivation of the conditional pdf of the phase, shows that it is Bennett's pdf, and extends this revealing (Theorem 1) to the multivariate case. In Section 4, we use the results examining the conditional pdf of the phase for the locked and unlocked reference phases and evaluate the error probability for the phase to exceed a threshold. Finally, concluding remarks are drawn in Section 5.

## 2. Signal model and problem formulation

Consider a received nonstationary random RF impulse signal

$$x(t)=\sqrt{2S}\alpha(t)\cos[2\pi f_0 t+\vartheta(t)], \tag{1}$$

where $U(t)=\sqrt{2S}\alpha(t)$ is an amplitude, $2S$ is a peak power, and $f_0$ is a carrier frequency. Here $\alpha(t)$ is a normalized waveform affected by fading and thereby making $U(t)$ nonstationary and randomly valued, $\vartheta(t)$ is an informative time-varying modulo $2\pi$ random phase[2] affected by the medium and noise in the carrier. In applications of (1) to phase systems, of interest are mostly the phase (or time delay), linear phase drift rate (Doppler), and linear frequency drift rate (Doppler rate) [2].

At the receiver, (1) is perturbed by the narrowband Gaussian noise

$$\begin{aligned} v(t)&=A(t)\cos[2\pi f_0 t+\varphi(t)]\\ &=A_c(t)\cos 2\pi f_0 t - A_s(t)\sin 2\pi f_0 t, \end{aligned} \tag{2}$$

where $A_c(t)=A(t)\cos\varphi(t)$ and $A_s(t)=A(t)\sin\varphi(t)$ are orthogonal, low-pass, stationary, and zero-mean Gaussian processes with known the one-sided power spectral density $S_v(f)$ that, by the band-pass filter, becomes symmetric about $f_0$. In causal systems, $v(t)$ is generated to have finite the multiple time derivatives.

An additive mixture of (1) and (2) yields the resulting signal

$$\begin{aligned} \xi(t)&=x(t)+v(t)=V(t)\cos[2\pi f_0 t+\theta(t)]\\ &=V(t)\cos\theta(t)\cos 2\pi f_0 t\\ &\quad - V(t)\sin\theta(t)\sin 2\pi f_0 t\\ &=[U(t)\cos\vartheta + A_c(t)]\cos 2\pi f_0 t\\ &\quad -[U(t)\sin\vartheta + A_s(t)]\sin 2\pi f_0 t \end{aligned} \tag{3}$$

provided the zero-mean Gaussian variables

$$A_c(t)=V(t)\cos\theta(t)-U(t)\cos\vartheta(t),$$

$$A_s(t)=V(t)\sin\theta(t)-U(t)\sin\vartheta(t), \tag{4}$$

with the equal variances $\sigma^2=\sigma_c^2=E\{A_c^2(t)\}=\sigma_s^2=E\{A_s^2(t)\}$. A mixture (3) is illustrated in Fig. 1 for the three typical waveforms [21]. If $U(t)$ is rectangular (Fig. 1a), all its time derivatives at $t_0$ are zero. Namely this case had been a subject for investigations in [4–14,16–18,22,24,29–31] and has become fundamental for digital communications. Contrary, all even time derivatives of the Gaussian waveform (Fig. 1b) used in pulse systems exist at $t_0$. Moreover, all the time derivatives of the triangular waveform (Fig. 1c) do not exist at $t_0$ at all. The picture is sophisticated if $\vartheta(t)$ has the nonzero time derivatives at $t_0$.

The problem now formulates as follows. Given the Gaussian variables (4), we would like to derive the conditional pdf of $\theta \bmod 2\pi$ allowing for the amplitude $U(t)$ and phase $\vartheta(t)$ to have the nonzero multiple time derivatives at $t_0$.

## 3. An exhaustive conditional pdf of the phase

Pursuing the aim, we shall first derive the conditional pdf of $\theta$ via the six-variable pdf of $A_c$ and $A_s$ and their first and second time derivatives. The result will then be generalized to the multivariate case.

[1] Gatkin et al. showed in [19] that, in contrast, the distribution of the instantaneous frequency is affected by the first time derivatives of the amplitude and phase.

[2] Throughout the paper we consider the modulo $2\pi$ phase and phase difference existing from $-\pi$ to $\pi$.