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Convolution and correlation theorems for the offset fractional Fourier transform and its application

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ABSTRACT

This paper presents new convolution and correlation theorems in the OFRFT domain. The authors also discuss the design method of multiplicative filter for bandlimited signals for OFRFT by convolution in time domain based on fast Fourier transform (FFT) as well as in OFRFT domain. Moreover, with the help of simulation, the effect of time-shifting and frequency-modulation parameters is shown in mapping one shape of an area to the same shape of another area.

1. Introduction

The offset fractional Fourier transform (OFRFT) [1,2] is similar to original fractional Fourier transform (FRFT) except for the presence of two extra parameters m and n, which corresponds to time (space) shifting and frequency-modulation respectively. It is a useful mathematical tool and is widely used for spectrum analysis, signal processing and optical system analysis. It is more flexible than FRFT for its two extra parameters. Well-known transforms such as the Fourier transform (FT) [3–5], the offset FT [1,2], the FRFT [6–12], time-shifting and scaling, frequency-modulation and others can be seen as the special cases of the OFRFT. Thus, by developing convolution and correlation theorems for OFRFT, unified convolution and correlation theorems for all of the above mentioned transforms are obtained. Convolution and correlation operations are fundamental in the theory of linear time-invariant (LTI) system [10,13]. The output of any continuous time LTI system is found via the convolution of the input signal with the system impulse response. In other words, Fourier transform of convolution of two signals is the point wise product of Fourier transform of the respective signals. Correlation, which is similar to convolution, is another important operation in signal processing, as well as in optics, in pattern recognition, especially in detection applications [14–16]. The correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed [17]. Convolution and correlation operations for FRFT have been proposed some years ago [17–31]. However, convolution and correlation operations for OFRFT still yet remain unknown. To be specific, the convolution theorem of the FT for the signals $f(t)$ and $g(t)$ with associated FTs, $F(u)$ and $G(u)$, respectively is given by:

$$f(t) \otimes g(t) \overset{FT}{\leftrightarrow} F(u)G(u) \tag{1}$$

where '$\otimes$' denotes the conventional convolution operation i.e.

$$f(t) \otimes g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \, d\tau \tag{2}$$

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This theorem elaborates a powerful result that the convolution of two signals in time domain leads to simple multiplication of their FTs in the frequency domain. The convolution theorem for the FRFT given in [20] lacks the relation given by (1), and the result from convolution operation is given by,

$$f(t)\otimes g(t)\overset{FRFT}{\leftrightarrow}\frac{|csc\alpha|}{\sqrt{2\pi}}\exp\left[j\left(\frac{u^2}{2}\right)\cot\alpha\right]\times\int_{-\infty}^{\infty}X_\alpha(v)\exp\left(-j\frac{v^2}{2}\tan\alpha\right)g\left[(u-v)\sec\alpha\right]dv \quad (3)$$

where, X_α denotes the FRFT of the signal $f(t)$. In 1998, Zayed [21] and in 2006, Deng et al. [22], proposed different definitions of convolution theorem in FRFT domain and these required three chirp multiplications to evaluate the defined convolution integral. The documented convolution operation is defined by

$$f(t)\otimes g(t)\overset{FRFT}{\leftrightarrow}\exp\left(-\frac{j}{2}u^2\cot\alpha\right)X_\alpha(u)Y_\alpha(u) \quad (4)$$

After three years in 2009, Wei et al. [23] derived a new expression for convolution theorem in FRFT domain but the generalized convolution operation defined in time domain is not only dependent on time variable but it also depends on transform domain variable 'u' in which it has to be transformed. The defined convolution operation is given by

$$\int_{-\infty}^{\infty}x(\tau)y(t\theta\tau)d\tau\overset{FRFT}{\leftrightarrow}X_\alpha(u)Y_\alpha(u) \quad (5)$$

where, $y(t\theta\tau)$ defines a τ-generalized translation of signal $y(t)$. Although the convolution theorems derived by Wei et al. [25] and Shi et al. [28,29] results in FT conversion at $\alpha=\pi/2$ but the multiplicative filter in FRFT domain is not realizable due to the presence of FT in the transform domain. Finally, Shi et al. [30,31] proposed generalized convolution theorems for FRFT and LCT in the year 2012 in which the convolution theorems derived independently in the literature [21,28,22,24,27] are the special cases of these generalized theorems. Although the results of the convolution theorem derived in [27] are consistently matching with the results of the proposed convolution theorem, but in the rest part of the paper, additional to derived results of the convolution theorem, the authors have derived the corresponding product theorem, correlation theorems as well as presented the simulation comparison. Moreover, the authors also have proved that computational complexity of the multiplicative filter in OFRFT domain with convolution in time domain that can be reduced to the computational complexity of the FFT as well as an application of multiplicative filtering in OFRFT domain is also presented. Moreover, the convolution and correlation theorems derived in [24,26] results in FT conversion at $\alpha=\pi/2$ as well as are expressible by a one dimensional integral and easy to implement. To validate the results of the proposed theorems, a simulation comparison is tried to establish between them. The rest of the paper is organized as follows: Section 2 gives the brief review to the OFRFT and its special cases. Convolution and correlation theorems for OFRFT are derived in Sections 3 and 4. In Section 5, a practical method to achieve the multiplicative filtering in the time domain is proposed based on the model of multiplicative filtering in the OFRFT domain as well as with the help of examples, results of filtering the signal in OFRFT domain using the proposed theorems are also presented. Simulation results and the effect of time-shifting and frequency-modulation parameters are presented in Section 6 and finally the conclusion is given in Section 7.

2. The offset FRFT

The OFRFT allows shifting/translation, rotation and squeezing of a signal to fit within a fixed window as compared to only rotation in case of FRFT. The integral form of one-dimensional OFRFT with (α, m, n) parameters of a signal $f(t)$ is defined [1] as:

$$O_F^{(\alpha,m,n)}(f(t))(u)=F(u)=\sqrt{\frac{1-j\cot\alpha}{2\pi}}\int_{-\infty}^{+\infty}f(t)K_{(\alpha,m,n)}(t,u)dt \quad (6)$$

where, parameters (α, m, n) are real numbers and $\alpha=a\pi/2$ is the rotation angle. a is the order of the OFRFT, when $a\in[0, 1]$, then $\alpha\in\left[0, \pi/2\ \right]$ and a is not equal to multiples of 2. The term $K_{(\alpha,m,n)}(u,t)$ represents the integral kernel and is given by

$$K_{(\alpha,m,n)}(t,u)=K_A\exp\left[\frac{j}{2}\left\{\left(u^2+t^2\right)\cot\alpha+2t(m-u)csc\alpha+2u(n-m\cot\alpha)\right\}\right] \quad (7)$$

where, $K_A=\exp\left(\frac{j}{2}m^2\cot\alpha\right)$. The integral form of inverse operation of the OFRFT with parameters (α, m, n) is given by

$$f(t)=\sqrt{\frac{(j\cot\alpha-1)}{2\pi}}\int_{-\infty}^{+\infty}F(u)K^*_{(\alpha,m,n)}(t,u)du \quad (8)$$

where $K^*_{(\alpha,m,n)}(t,u)$ indicates the complex conjugate of the integral kernel. The two basic properties of OFRFT are the additivity and reversibility properties [1].

Property 1. *Additivity property*

$$O_F^{(\alpha_2,m_2,n_2)}\left[O_F^{(\alpha_1,m_1,n_1)}f(t)\right]=\exp(j\varphi)O_F^{(\alpha_1+\alpha_2,m_3,n_3)}[f(t)] \quad (9)$$

where

$$m_3=m_2+m_1\cos\alpha_1+n_1\sin\alpha_1 \quad (10)$$

$$n_3=n_2-m_1\sin\alpha_1+n_1\cos\alpha_1 \quad (11)$$

$$\varphi=\frac{\sin(2\alpha_2)}{4}\left(m_1^2-n_1^2\right)+m_1n_1\sin^2\alpha_2+(m_1\sin\alpha_2-n_1\cos\alpha_2)m_2 \quad (12)$$