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The solids-flux theory – Confirmation and extension by using partial differential equations

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ABSTRACT

The solids-flux theory has been used for half a century as a tool for estimating concentration and fluxes in the design and operation of secondary settling tanks during stationary conditions. The flux theory means that the conservation of mass is used in one dimension together with the batch-settling flux function according to the Kynch assumption. The flux theory results correspond to stationary solutions of a partial differential equation, a conservation law, with discontinuous coefficients modelling the continuous-sedimentation process in one dimension. The mathematical analysis of such an equation is intricate, partly since it cannot be interpreted in the classical sense. Recent results, however, make it possible to partly confirm and extend the previous flux theory statements, partly draw new conclusions also on the dynamic behaviour and the possibilities and limitations for control. We use here a single example of an ideal settling tank and a given batch-settling flux in a whole series of calculations. The mathematical results are adapted towards the application and many of them are conveniently presented in terms of operating charts.

1. Introduction

The complexity of the secondary settling tank (SST) within the activated sludge process is not only known to operators of the plants, but also to researchers. Interesting nonlinear phenomena are shown even if simplifying idealizations are made in models. The same experiences have been made in other fields, since clarifier–thickener units are also used in the mineral, chemical, pulp-and-paper and food industries.

Several simulation models have been suggested for clarifier–thickener units. This has been a natural development, particularly for the SST, because of its presence in the activated sludge process and the fact that there are reliable simulation models for the biological reactor.

However, even if reliable simulation programs are developed, they do not give general rules on how to control the process. Then one has to go back and investigate properties of the basic physical laws that govern the process. Such a law is the conservation of mass, which can be written exactly as an equation with integrals. Such a continuity equation can be reformulated and written equivalently as a partial differential equation (PDE), also called a conservation law, provided that it is interpreted in the so-called *weak sense*. This is a mathematical terminology, which means that *the PDE is only a convenient symbol for an equation containing integrals*. For example, a solution of a PDE may have discontinuities, despite the fact that such are not differentiable in the ordinary sense. This sometimes causes misunderstandings. Conclusions, or numerical algorithms, derived directly from the PDE may thus be incorrect.

Although different ways of tackling the problem have been presented during half a century by water researchers, chemical engineers, applied mathematicians, etc., a common platform has been the celebrated paper by Kynch (1952), which we

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Nomenclature

A	cross-sectional area [m^2]
D	depth of thickening zone [m]
$\mathcal{D}$	'dangerous' region in operating chart
$\mathcal{E}$	excess flux in steady state [kg/(m^2 h)]
F	total flux function in Eq. (1) [kg/(m^2 h)]
H	height of clarification zone [m]
$\mathcal{O}$	overloaded region in operating chart
Q	volumetric flow rate [m^3/h]
Q_u^{max}	maximum bound on control variable, cf. Eq. (10) [m^3/h]
Q_u^{min}	minimum bound on control variable, cf. Eq. (11) [m^3/h]
$\mathcal{S}$	'safe' region in operating chart
$\mathcal{U}$	underloaded region in operating chart
X	total suspended solids concentration [kg/m^3]
X_{infl}	inflection point of flux functions [kg/m^3]
X_{max}	maximum concentration [kg/m^3]
X^M	local maximum point of f [kg/m^3]
X_M	local minimum point of f [kg/m^3]
X_m	concentration satisfying $X_m \leq X_{infl}$ and $f(X_m) = f(X_M)$ [kg/m^3]
X_u^{min}	minimum bound on underflow concentration, cf. Eq. (9) [kg/m^3]
f	flux function in thickening zone [kg/(m^2 h)]
f_b	batch-settling flux function [kg/(m^2 h)]
f_{lim}	limiting flux function [kg/(m^2 h)]
f_{thick}	flux in thickening zone in steady state [kg/(m^2 h)]
ℓ	line in operating chart
p	point in operating chart
q	volumetric flux [m/h]
s	source term, flux fed to SST [kg/(m^2 h)]
t	time [h]
v_{settl}	settling velocity [m/h]
x	depth from feed level [m]
y	flux axis in operating chart [kg/(m^2 h)]

Greek letters

Λ	region in steady-state control chart
δ	Dirac delta distribution [1/m]
θ	X_u/X_f thickening factor [–]

Subscripts

0	initial value at $t = 0$
cl	clarification zone
e	effluent
f	feed (inlet)
th	thickening zone
u	underflow

Superscripts

$=$	related to inflection point
$\sim$	related to critically loaded SST
$\wedge$	related to optimal operation

may consider to be the origin of the *solids-flux theory*. Kynch's constitutive assumption is that the local settling velocity is a function of the concentration only, $v_{settl}(X)$, and it is decreasing. Batch sedimentation in a column can in one dimension be described by the conservation law (interpreted in the weak sense)

$$\frac{\partial X}{\partial t} + \frac{\partial f_b(X)}{\partial x} = 0,$$

where $f_b(X) = Xv_{settl}(X)$ is the batch-settling flux function. It is assumed that this function has an inflection point and a typical graph is shown in Fig. 1. Kynch also showed how solutions could be constructed by the method of characteristics.

Fundamental results of graphical constructions by using the flux curve $f_b(X)$ for obtaining concentrations in steady-state operation were presented by Jernqvist (1965a–c). Unfortunately, Jernqvist's results seem not to have reached other researchers. Similar developments, but not as extensive, were presented in the 1960s onwards with concepts such as the operating line, the limiting flux and the state point (pivot point, feed point), see e.g. Ekama et al. (1997), Diehl (2001) and references therein.

Interpretations concerning clarification failure and control were made by Keinath et al. (1977); Laquidara and Keinath (1983); Keinath (1985) and a chart describing steady states was presented by Lev et al. (1986). Later references also showed the need for the flux theory for describing steady states of the process (e.g. Chancelier et al., 1997a; Ekama and Marais, 2004; Kaushik and Murthy, 2002; Lynggaard-Jensen and Lading, 2006; Narayanan et al., 2000; Wett, 2002; Wilén et al., 2004).

At the same time, one should be aware of the inherent limitations of a one-dimensional ideal model together with only one constitutive assumption (Kynch's). The complexity of two- and three-dimensional models makes, however, a one-dimensional model still interesting for the design of full-scale SSTs. Ekama and Marais (2004) then recommend a safety factor (reduction factor) to account for the hydraulic non-idealities.

A first-order PDE model of the entire SST was presented independently by Chancelier et al. (1994) and Diehl (1995, 1996). It is a one-dimensional model together with Kynch's constitutive assumption. Hence, *solutions of the PDE are physically correct solutions within the flux theory*. Consequently, the flux theory can be confirmed and extended in a natural way within the context of PDE theory. This was done by Chancelier et al. (1997a,b). It is the aim of the present paper to present further results in this direction. We emphasize that 'a physically correct solution' refers to the physical law, or equivalently, the PDE and the constitutive assumption. To what extent such a solution approximates the real concentrations of an SST is another issue not dealt with here.

The inlet and outlets of the SST cause differing fluxes and conditions, which imply that coefficients in the PDE are space discontinuous. These discontinuities imply a severe nonlinear feature in addition to the nonlinear sedimentation–consolidation process itself. This has implied that basic research in mathematics has had to be carried out in order to obtain a correct description of the process and to develop reliable numerical methods, see the special issue on conservation laws with discontinuous flux with the editors Bürger and Karlsen (2008).