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Combining triangle Gaussian integration and modified NUFFT for evaluating two-dimensional Fourier transform integrals

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ABSTRACT

The regular fast Fourier transform (FFT) requires a uniform Cartesian orthogonal grid which has considerable stair-casing errors when dealing with the function having an arbitrary shape boundary. The recently proposed two-dimensional discontinuous fast Fourier transform (2D-DFFT) can overcome this problem by using triangle mesh discretization and Gaussian numerical integration. However, the interpolation is used for the function data in the original 2D-DFFT, which reduces the accuracy performance especially for the case of oscillating functions. This work presents a useful modification of the original 2D-DFFT by removing the requirement of function interpolation to obtain significant accuracy improvement. In addition, the modified 2D nonuniform fast Fourier transform (NUFFT) with real-valued least-square interpolation coefficients are developed to speed up the computation of numerical Fourier transform over the triangle mesh. Numerical experiments are conducted to demonstrate the effectiveness and advantages of the proposed algorithms.

1. Introduction

The Fourier transform method has many applications such as the evaluation of the secondary pattern from an aperture field [1,2], solution to electromagnetic scattering problems [3–5], and fast measurement of stress anisotropy in magnetic thin films [6]. In these applications, the functions to be transformed may have discontinuities at the boundaries or at the interfaces of multilayered material. Such discontinuous boundaries usually have arbitrary shapes, and consequently the regular fast Fourier transform (FFT) algorithm based on an orthogonal grid has considerable stair-casing error for approximating the continuous Fourier transform. In very few special cases, closed-form solutions are available for the Fourier transform of 2D constant [2,7] or linear distributions [8] with polygonal shape boundaries. Naturally, they cannot evaluate directly the Fourier transform of an arbitrary distribution. Several fast numerical Fourier transform methods have been proposed previously for the case of one-dimensional (1D) discontinuous functions [9–11]. Among them, the discontinuous fast Fourier transform (DFFT) algorithm presented in [11] has been recently extended to the two-dimensional (2D) case [12]. Either the 1D or 2D-DFFT algorithm employs a double interpolation procedure: one is for interpolating the function and the other is for the exponential. In particular, the 2D-DFFT algorithm in [12] uses triangle mesh to discretize the support domain of the function, and applies the Gaussian numerical integration over the triangular mesh to perform the Fourier transform [13]. Such processing makes the discretization grid capable of conforming with any boundary shape, and consequently removes the stair-casing approximation error. This technique leads to the requirement of evaluating the 2D nonuniform discrete Fourier transform (2D-NUDFT) which can be efficiently done by using the nonuniform fast Fourier transform (NUFFT) algorithm presented in [14,15].

It should be noted that the 2D-DFFT presented in [12] was developed for potential applications in solving Maxwell's integral equations [3–5]. To guarantee the continuity of physical variables (e.g., the electrical current continuity), the function is assumed to be sampled only at the triangle vertexes, and values at the Gaussian integration points are obtained by linear interpolation of the sampled data. However, the linear interpolation degrades the accuracy performance of 2D-DFFT. Especially for the oscillating functions, its accuracy will be significantly reduced. To our knowledge, there are also many situations where the sampling positions can be chosen according to the computational accuracy and efficiency, and the interpolation of function is unnecessary. For example, the pseudo-spectral time-domain (PSTD) method, well known for its capability of solving large-scale electromagnetic or acoustic problems [16], needs to evaluate the direct Fourier transform of physical components at a set of prefixed positions. Due to the limitation of regular FFT, the PSTD is usually applied to the uniform grid. Actually we can choose Gaussian integration points as the prefixed grid for the Fourier transform (this will extend the original PSTD method to

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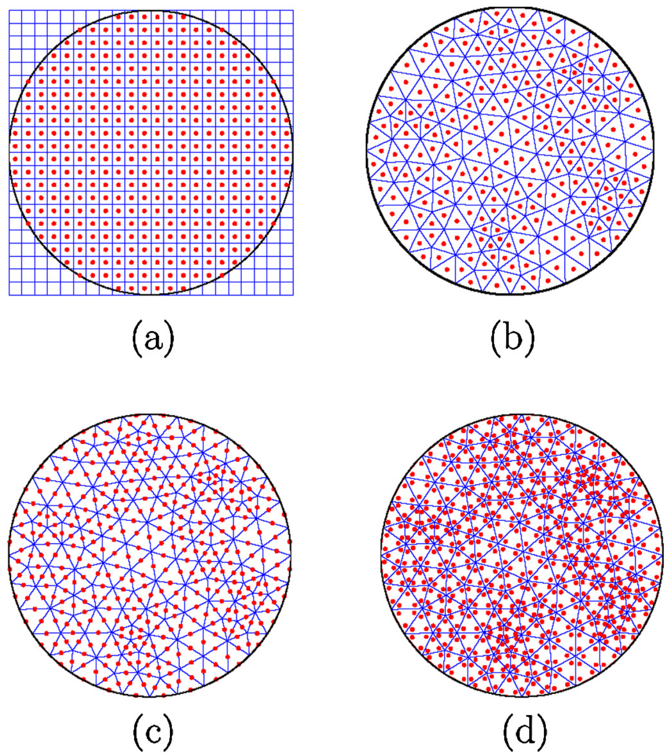

Fig. 1. Sampling patterns of a circular plane for FFT (a) and for the proposed algorithms with Gaussian integration schemes I (b), II (c), and III (d).

suit more general geometry). In this case, no interpolation of the function is required.

Here the 2D-DFFT is modified for directly evaluating the Fourier transform of a 2D discontinuous function. The proposed modifications include: (a) the function is sampled at the Gaussian integration points instead of at the triangle vertexes in the original 2D-DFFT, and no interpolation is required for the function data; (b) the 1D modified NUFFT presented in [17] is here extended to the 2D case, and the modified 2D-NUFFT has real-valued interpolation coefficients to halve the storage and computational cost of the kernel convolution process required by the NUFFT. Although the original 2D-DFFT can guarantee the continuity of the function across adjacent triangle elements which is required for some particular applications, the modified algorithm is much more accurate and efficient, especially for the case of oscillating functions. A number of numerical experiments are conducted to verify the performance and advantage of the modified algorithm.

2. Triangle discretization and Gaussian integration

The problem of finding the Fourier transform of a 2D discontinuous function can be reduced to the evaluation of the following integral:

$$F(\mathbf{k}) = \sum_i \int_{S_i} f(\mathbf{x}) e^{-j\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \tag{1}$$

where $\mathbf{x} = [x, y]$, $\mathbf{k} = [k_x, k_y]$, and $f(\mathbf{x})$ is assumed to be continuous inside the individual support domain S_i.

The regular FFT requires a uniform orthogonal grid to discretize the support domain. For example, Fig. 1(a) shows the sampling pattern of 2D FFT for a circular surface. As can be seen, the sampling with an orthogonal grid cannot capture well the boundary shape of the support. To overcome this problem, we use triangular elements to discretize the support. The discretization with triangles has the ability to conform with an arbitrary shape boundary. Symmetrical Gaussian quadrature formulas over the triangles [13] are then adopted to approximate the integral of (1). Here, three Gaussian quadrature formulas with different quadrature nodes and weights are used for comparison. The resulting three sampling patterns are shown in Fig. 1(b), (c) and (d), respectively. The nodes and weights for each kind of Gaussian quadrature [13] are shown in Table 1. Accordingly, the integral of (1) can be approximately calculated by collecting the Gaussian quadratures over all the triangles. That is,

Table 1
Gaussian quadrature weights and nodes corresponding to three sampling schemes shown in Fig. 1(b)–(d).

	w_i	α_i	β_i	γ_i
Scheme I	1	1/3	1/3	1/3
Scheme II	1/3	1/2	1/2	0
	1/3	0	1/2	1/2
	1/3	1/2	0	1/2
Scheme III	1/3	2/3	1/6	1/6
	1/3	1/6	2/3	1/6
	1/3	1/6	1/6	2/3

$$\int_S f(\mathbf{x}) e^{-j\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \approx \sum_{l=1}^{L} A^l \sum_{i=1}^{I} w_i f(\mathbf{x}_i^l) e^{-j\mathbf{k}\cdot\mathbf{x}_i^l} \tag{2}$$

where A^l denotes the area of the lth triangle, w_i denotes the weight of the i-th node, and $\mathbf{x}_i^l$ denotes the location of the ith node of the lth triangle. $\mathbf{x}_i^l$ can be calculated by

$$\mathbf{x}_i^l = \alpha_i \mathbf{x}_a^l + \beta_i \mathbf{x}_b^l + \gamma_i \mathbf{x}_c^l \tag{3}$$

where $\mathbf{x}_a^l$, $\mathbf{x}_b^l$ and $\mathbf{x}_c^l$ denote the locations of three vertexes of the lth triangle, and the definitions of α_i, β_i and γ_i are given in Table 1. Here we assume that the sampling positions used for numerical Fourier transform are put exactly on the Gaussian points. That is, the value $f(\mathbf{x}_i^l)$ is already known and no function interpolation is needed. This is different from the original 2D-DFFT in [12] where the function is sampled at the triangle vertexes and the values at Gaussian points are obtained by linear interpolation.

We can rewrite (2) as in the form of Fourier summation

$$F(\mathbf{k}) = \sum_{m=0}^{M-1} H_m e^{-j\mathbf{k}\cdot\mathbf{x}'_m} \tag{4}$$

where $M = LI$, $H_{(l-1)I+i-1} = A^l w_i f(\mathbf{x}_i^l)$, and $\mathbf{x}'_{(l-1)I+i-1} = \mathbf{x}_i^l$. Sampling the above expression in $\mathbf{k}$ space with $\mathbf{k} = [2\pi\Delta_1 n_1, 2\pi\Delta_2 n_2]$ where $n_1 = -N_1/2 \ldots N_1/2 - 1$ and $n_2 = -N_2/2 \ldots N_2/2 - 1$, we obtain

$$F_{\mathbf{n}} = \sum_{m=0}^{M-1} H_m e^{-j2\pi\mathbf{v}_m\cdot\mathbf{n}} \tag{5}$$

where $\mathbf{n} = [n_1, n_2]$ and $\mathbf{v}_m = [\Delta_1, \Delta_2].*\mathbf{x}'_m$. The symbol ".*" denotes the element-by-element multiplication. Clearly, (5) is in the form of 2D NUDFT.

3. Modified 2D-NUFFT with real-valued interpolation coefficients

As is well known, direct evaluation of (5) costs $O(MN_1N_2)$ operations. Recent developments in nonuniform fast Fourier transform (NUFFT) algorithms [15,17,18] provide more efficient tools to evaluate the NUDFT. In particular, the modified NUFFT algorithm that has real least-square (LS) interpolation coefficients is recently presented in [17], and this algorithm achieves the reduced computation and storage cost compared with the original NUFFT. The