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Improved harmonic balance implementation of Floquet analysis for nonlinear circuit simulation

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ABSTRACT

We present a novel algorithm for the efficient numerical computation of the Floquet quantities (eigenvalues, direct and adjoint eigenvectors) relevant to the assessment of the stability and noise properties of nonlinear forced and autonomous circuits. The approach is entirely developed in the frequency domain by means of the application of the Harmonic Balance technique, thus avoiding lengthy time–frequency transformations which might also impair the accuracy of the calculated quantities. An improvement in the computation time around one order of magnitude is observed.

1. Introduction

Floquet analysis, although a classical topic in the mathematical literature [1,2], is the object of a renewed interest in the circuit simulation community because of the central role played in two important areas of nonlinear circuit performance assessment: the rigorous study of phase and amplitude noise in oscillators [3–8], and the appraisal of the stability of the time-periodic working point solution of a nonlinear circuit, either driven or autonomous [1,2,9–13].

Floquet theorem was originally derived with reference to Ordinary Differential Equations (ODEs) [1], and recently has been extended to the case of index-1 Differential Algebraic Equations (DAEs) [14]. This step has significant practical importance, since DAE is the general form of the describing equations obtained from the application of nodal analysis to a nonlinear circuit made of lumped components [15]. In general, the circuit nodal equations can be cast in the form

$$\begin{bmatrix} \boldsymbol{L}_1 \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f}(\boldsymbol{x}(t)) \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_2\boldsymbol{x} \\ \boldsymbol{g}(\boldsymbol{x}(t), t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{c}(t) \\ \boldsymbol{d}(t) \end{bmatrix} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the unknown vector, $\boldsymbol{L}_1, \boldsymbol{L}_2 \in \mathbb{R}^{m\times n}$ with $m \leq n$ are constant matrices describing the linear part of the circuit, $\boldsymbol{c}(t) \in \mathbb{R}^m$ and $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{d} \in \mathbb{R}^{n-m}$. The nonlinear functions $\boldsymbol{f}$, $\boldsymbol{g}$ are due to the nonlinear elements in the circuit. The forcing terms in (1) (if any) are time-periodic functions of period T, and a non-trivial T-periodic solution (limit cycle) $\boldsymbol{x}_S(t)$ is assumed to exist.[1]

Floquet theorem [14] applies to the linearization of (1) around $\boldsymbol{x}_S(t)$, i.e. it is a tool to characterize the effect of a small-change perturbation applied to the circuit limit cycle (see [11], and references therein). The main result is that the stability of $\boldsymbol{x}_S(t)$ is dependent on a set of n (complex) numbers λ_k ($k=1, \ldots, n$) called the *Floquet multipliers* (FMs) of $\boldsymbol{x}_S(t)$: if all of them (or, for oscillators, all but one which is exactly equal to 1) are placed strictly inside the unit circle of the complex plane, the limit cycle is asymptotically stable; on the other hand, if at least one of them has magnitude larger than one, the solution is unstable. The computation of the FMs provides therefore an assessment of the stability of the circuit working point. On the other hand, each FM is also associated to a *(direct) Floquet eigenvector* $\boldsymbol{u}_k(t)$ ($k=1, \ldots, n$) which, together with the *adjoint Floquet eigenvectors* $\boldsymbol{v}_k(t)$ ($k=1, \ldots, n$) associated to the adjoint linearized system[2], are the basic ingredients required to perform oscillator noise analysis [3,7,8]

The calculation of the Floquet multipliers and eigenvectors can be performed either in the time or in the frequency domain. The problem has been traditionally tackled for ODEs in the time domain [9,11,16,17], devising numerical approaches with various degrees of efficiency and accuracy. On the other hand for many applications spectral techniques, such as the harmonic balance (HB) method,

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[1] The same hypothesis holds for the case of autonomous circuits, for which of course no source term is present.
[2] Both the direct and adjoint linearized systems share the same FMs [1,14].

provide significant advantages for the determination of the circuit limit cycle [15,18,19]. A general purpose, frequency-domain algorithm based on the HB approach for the determination of the direct and adjoint Floquet quantities was proposed in [10] and [20], respectively. In both cases, a generalized eigenvalue problem has to be solved, thus requiring a numerical procedure whose computational burden is $O(N^3)$ where N is the matrices size. Notice that for HB, $N = n(2N_\mathrm{H}+1)$ where N_H is the number of harmonics included in the simulation besides DC. In this contribution, we propose a numerical approach applicable to both the direct and adjoint problem which, making use of fast matrix manipulation completely taking place in the frequency domain, allows for a significant advantage with respect to previous algorithms by reducing the computational complexity to $O(N^2)$. The algorithm ultimately corresponds to a general methodology for constructing a linear ODE fully equivalent to the linearized DAE, thus extending the applicability of the approach in [9,16,17].

2. Fundamentals

We provide here the fundamentals required to effectively present the numerical procedure we developed. Linearization of (1) around the limit cycle $\boldsymbol{x}_\mathrm{S}(t)$ leads to a Linear Periodic Time Varying (LPTV) system of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{C}(t)\boldsymbol{z}] = \boldsymbol{A}(t)\boldsymbol{z}(t) \tag{2}$$

where $\boldsymbol{C}, \boldsymbol{A} \in \mathbb{R}^{n\times n}$ are T-periodic matrices (since they correspond to the Jacobian of the full system calculated into $\boldsymbol{x}_\mathrm{S}(t)$) and $\boldsymbol{z}(t) \in \mathbb{R}^n$ represents the perturbation of the limit cycle. In many cases, matrix $\boldsymbol{C}(t)$ is not full rank although we assume that the rank $\rho \le n$ is independent of time (index-1 DAE) [14]: this has important consequences on the calculation of the Floquet quantities, as we will see shortly.

According to the generalization of Floquet theorem to DAEs [14], (2) is solved by ρ independent functions taking the form $\boldsymbol{z}(t) = \exp(\mu_k t)\boldsymbol{u}_k(t)$ $(k = 1, \ldots, \rho)$, where $\mu_1, \ldots, \mu_\rho$ are the *Floquet exponents* (FEs) of (2) (and $\lambda_k = \exp(\mu_k T)$ are the corresponding *Floquet multipliers*), $\boldsymbol{u}_k(t)$ is T-periodic and is called the *direct Floquet eigenvector* associated to μ_k. Notice that $\rho < n$ corresponds to the appearance of $n-\rho$ FEs equal to $-\infty$.

On the other hand, the *adjoint* system to (2) reads [14]

$$\boldsymbol{C}^\mathrm{T}(t)\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = -\boldsymbol{A}^\mathrm{T}(t)\boldsymbol{w}(t) \tag{3}$$

and is solved by a linear combination of the ρ independent functions $\boldsymbol{w}(t) = \exp(-\mu_k t)\boldsymbol{v}_k(t)$, where $\boldsymbol{v}_k(t)$ is again T-periodic and is called the *adjoint Floquet eigenvector* associated to μ_k.

2.1. The HB technique

The HB technique is based on representing each scalar time periodic function $\alpha(t)$ through the (truncated) Fourier series

$$\alpha(t) = \tilde{\alpha}_{0,\mathrm{c}} + \sum_{h=1}^{N_\mathrm{H}} \left[\tilde{\alpha}_{h,\mathrm{c}}\cos(h\omega_0 t) + \tilde{\alpha}_{h,\mathrm{s}}\sin(h\omega_0 t)\right] \tag{4}$$

where $\tilde{\alpha}_{0,\mathrm{c}}$ represents the DC component of $\alpha(t)$. The harmonic components are collected into a vector of size $2N_\mathrm{H}+1$ $\tilde{\boldsymbol{\alpha}} = [\tilde{\alpha}_{0,\mathrm{c}}, \tilde{\alpha}_{1,\mathrm{c}}, \tilde{\alpha}_{1,\mathrm{s}}, \ldots, \tilde{\alpha}_{N_\mathrm{H},\mathrm{c}}, \tilde{\alpha}_{N_\mathrm{H},\mathrm{s}}]^\mathrm{T}$, and put in one-to-one correspondence with a set of $2N_\mathrm{H}+1$ time samples (distributed into the interval $]0, T]$) of $\alpha(t)$, collected into vector $\hat{\boldsymbol{\alpha}} = [\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_{2N_\mathrm{H}+1})]^\mathrm{T}$. The relationship between $\hat{\boldsymbol{\alpha}}$ and $\tilde{\boldsymbol{\alpha}}$ is provided by an invertible linear operator $\boldsymbol{\Gamma}^{-1}$ corresponding to the discrete Fourier transform (DFT) [15]

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\alpha}} \Leftrightarrow \tilde{\boldsymbol{\alpha}} = \boldsymbol{\Gamma}\hat{\boldsymbol{\alpha}}. \tag{5}$$

Notice that the matrix representation is used for formal derivation only: in the actual implementation the more efficient DFT algorithm [15] is used.

Denoting as $\dot{\alpha}(t)$ the first derivative of $\alpha(t)$, trivial calculations yield the Fourier representation of the derivative as a function of the Fourier components of the original function

$$\tilde{\dot{\boldsymbol{\alpha}}} = \boldsymbol{\Gamma}\hat{\dot{\boldsymbol{\alpha}}} = \boldsymbol{\Omega}\tilde{\boldsymbol{\alpha}}, \tag{6}$$

where $\boldsymbol{\Omega} \in \mathbb{R}^{(2N_\mathrm{H}+1)\times(2N_\mathrm{H}+1)}$ is a tridiagonal costant matrix proportional to ω_0 (see [10,15] for the explicit representation).

In case of an n size vector $\boldsymbol{\alpha}(t)$, (5) and (6) are easily generalized by considering a vector of time-sample vectors and frequency components, defined by expanding each element $\alpha_j(t)$ $(j = 1, \ldots, n)$ into the time sample $\hat{\boldsymbol{\alpha}}_j$ and similarly for the harmonic components. One finds

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\Gamma}_n^{-1}\tilde{\boldsymbol{\alpha}} \qquad \tilde{\dot{\boldsymbol{\alpha}}} = \boldsymbol{\Omega}_n\tilde{\boldsymbol{\alpha}}, \tag{7}$$

where $\boldsymbol{\Gamma}_n^{-1}$ and $\boldsymbol{\Omega}_n$ are block diagonal matrices built replicating n times the fundamental operators $\boldsymbol{\Gamma}^{-1}$ and $\boldsymbol{\Omega}$, respectively.

More attention is required to derive the HB representation of $\boldsymbol{\beta}(t) = \boldsymbol{\Xi}(t)\boldsymbol{\alpha}(t)$ and of its time derivative, where $\boldsymbol{\Xi}(t)$ is a T-periodic matrix and $\boldsymbol{\alpha}(t)$ a T-periodic vector. Denoting as $\hat{\boldsymbol{\Xi}}$ the $n\times n$ block diagonal matrix built expanding each element $\xi_{h,k}(t)$ of $\boldsymbol{\Xi}(t)$ as a $(2N_\mathrm{H}+1)\times(2N_\mathrm{H}+1)$ diagonal matrix formed by the time samples $\hat{\xi}_{h,k}$, we have

$$\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\alpha}} \qquad \tilde{\dot{\boldsymbol{\beta}}} = \boldsymbol{\Omega}_n\tilde{\boldsymbol{\beta}} = \boldsymbol{\Omega}_n\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\alpha}} \tag{8}$$

where $\tilde{\boldsymbol{\Xi}} = \boldsymbol{\Gamma}_n\hat{\boldsymbol{\Xi}}\boldsymbol{\Gamma}_n^{-1}$. The transformation leading to $\tilde{\boldsymbol{\Xi}}$ results into the sum of a Toepliz and of a Hankel matrix (see [21] for details), whose building blocks are the Fourier coefficients of the elements of $\boldsymbol{\Xi}(t)$: in other words, $\tilde{\boldsymbol{\Xi}}$ can be easily assembled after evaluating (through DFT) the Fourier coefficients of the elements of $\boldsymbol{\Xi}(t)$.

3. Previous work

Substituting $\boldsymbol{z}(t) = \exp(\mu_k t)\boldsymbol{u}_k(t)$ into (2) and $\boldsymbol{w}(t) = \exp(-\mu_k t)\boldsymbol{v}_k(t)$ into (3), the Floquet direct and adjoint eigenproblems are made explicit in the time domain [10,20]

$$\mu_k\boldsymbol{C}(t)\boldsymbol{u}_k(t) = \boldsymbol{A}(t)\boldsymbol{u}_k(t) - \frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{C}(t)\boldsymbol{u}_k(t)] \tag{9}$$

$$\mu_k\boldsymbol{C}^\mathrm{T}(t)\boldsymbol{v}_k(t) = \boldsymbol{A}^\mathrm{T}(t)\boldsymbol{v}_k(t) + \boldsymbol{C}^\mathrm{T}(t)\frac{\mathrm{d}\boldsymbol{v}_k(t)}{\mathrm{d}t}. \tag{10}$$

After time-sampling, the use of (6) and (8) allows to convert (9) and (10) in the spectral domain

$$\mu_k\tilde{\boldsymbol{C}}\tilde{\boldsymbol{u}}_k = \left[\tilde{\boldsymbol{A}} - \boldsymbol{\Omega}_n\tilde{\boldsymbol{C}}\right]\tilde{\boldsymbol{u}}_k \tag{11}$$

$$\mu_k\tilde{\boldsymbol{C}}_\mathrm{T}\tilde{\boldsymbol{v}}_k = \left[\tilde{\boldsymbol{A}}_\mathrm{T} + \tilde{\boldsymbol{C}}_\mathrm{T}\boldsymbol{\Omega}_n\right]\tilde{\boldsymbol{v}}_k. \tag{12}$$

Notice that $\tilde{\boldsymbol{C}}_\mathrm{T} = \boldsymbol{\Gamma}_n\hat{\boldsymbol{C}}^\mathrm{T}\boldsymbol{\Gamma}_n^{-1}$ is not simply the transpose of $\tilde{\boldsymbol{C}}$ (the same holds for $\tilde{\boldsymbol{A}}_\mathrm{T}$). Nevertheless, since $\hat{\boldsymbol{C}}$ is made of diagonal blocks (deriving from the time-sampling of the elements of $\boldsymbol{C}(t)$), $\tilde{\boldsymbol{C}}_\mathrm{T}$ can easily be built from the components of $\tilde{\boldsymbol{C}}$ avoiding any further calculation [20]. In summary, the Floquet quantities can be calculated as the solution of the generalized eigenvalue problems in (11) and (12), whose matrices correspond to the Jacobians of the HB problem defining the limit cycle, and therefore are available as a byproduct of the Newton iterations normally exploited for the determination of $\boldsymbol{x}_\mathrm{S}(t)$.

The solution of the generalized eigenproblems (11) and (12) yields $n(2N_\mathrm{H}+1)$ FEs (and the corresponding direct and adjoint eigenvectors). In an ideal system, i.e. if the number of harmonics is large enough, the FEs should be positioned in the complex plane