Topological chaos of the spatial prisoner's dilemma game on regular networks

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HIGHLIGHTS

- Dynamics of frequency of cooperation is sensitive to the initial conditions.
- Frequency of cooperation could not be adopted by simply averaging.
- Spatial prisoner's dilemma game is chaos.
- Procedure is applicable to other deterministic spatial models therein.

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ABSTRACT

The spatial version of evolutionary prisoner's dilemma on infinitely large regular lattice with purely deterministic strategies and no memories among players is investigated in this paper. Based on the statistical inferences, it is pertinent to confirm that the frequency of cooperation for characterizing its macroscopic behaviors is very sensitive to the initial conditions, which is the most practically significant property of chaos. Its intrinsic complexity is then justified on firm ground from the theory of symbolic dynamics; that is, this game is topologically mixing and possesses positive topological entropy on its subsystems. It is demonstrated therefore that its frequency of cooperation could not be adopted by simply averaging over several steps after the game reaches the equilibrium state. Furthermore, the chaotically changing spatial patterns via empirical observations can be defined and justified in view of symbolic dynamics. It is worth mentioning that the procedure proposed in this work is also applicable to other deterministic spatial evolutionary games therein.

1. Introduction

Evolutionary game theory has been considered an important approach to characterizing and understanding the evolution of cooperation in many settings of ecology, infectious disease dynamics, animal behaviors and social interactions of humans (Maynard and Price, 1973; Maynard, 1982; May and Leonard, 1975; Bshary et al., 2008; Nowak and May, 1994; Nowak and Sigmund, 2004a; Nowak et al., 2002; Nowak and Sigmund, 2005; Axelrod and Hamilton, 1981). The traditional methodological perspective to evolutionary game theory is based on the deterministic dynamics and the replicator equation in infinitely large, well-mixed populations (Hofbauer and Sigmund, 1988, 2003). Both in animal and human societies, only a limited number of individuals are interacted with each other (Bin et al., 2010). Therefore, the spatial version of evolutionary games taking place on networks is proposed to comprehend the evolutionary dynamics and provide better explanations for the emergence of cooperation (Nowak and May, 1992; Nakamaru et al., 1997; Hauert and Doebeli, 2004; Hauert and Imhof, 2012).

Since the groundwork on repeated games by Axelrod and Dion (1988), the evolutionary prisoner's dilemma game as a general metaphor for studying the cooperative behavior has drawn much attention in both theoretical and experimental studies (Rapoport and Chaammah, 1965; Fudenberg and Maskin, 1990; Nowak and Sigmund, 1993; Boerlijst et al., 1997). A spatial prisoner's dilemma game (SPDG) on a square lattice consisting of a two-state cellular automata was first reported in (Nowak and May, 1992). Subsequently, a series of investigations of SPDG on regular lattices and

graphs are quickly rose to prominence, particularly in the manner of cellular automata (Nowak and Sigmund, 1993; Mar and Denis, 1993; Sigmund, 1994; Grim, 1995, 1996; Pereira et al., 2008a, 2008b; Marcelo and Alexandre, 2010; Schimit et al., 2015). Many interests have also been given to the effects of spatial structures, such as small-world networks, scale-free networks, inter-dependent networks, dynamical networks, hierarchical networks, coevolutionary networks, networks of networks, etc (Lieberman et al., 2005; Ohtsuki et al., 2006; Ohtsuki and Nowak, 2006b; Taylor et al., 2007; Cao and Li, 2008; Fu et al., 2009; Li and Cao, 2009; Ohtsuki et al., 2007a; Perc and Wang, 2010; Wang et al., 2011; Antal et al., 2009b; Tarnita et al., 2009a; Wu et al., 2010; Changbing et al., 2012). Due to the enormous number of possible configurations, the asymptotic behaviors and the crucial quantity (frequency of cooperation) of this game for various parameter values are widely adopted by averaging over several steps after the game reaches the equilibrium state, and inferred almost entirely based on exhaustive computer simulations. The long-term time-asymptotic dynamics of frequency of cooperation studied in this work, however, is very sensitive to the initial conditions, rendering long-term prediction impossible in general. Meanwhile, the rigorous analysis of chaotically changing spatial patterns never-theless is missed in the framework of symbolic dynamics. None-theless, it has been well recognized that network topologies can play a crucial role in the evolution of cooperation, in addition to the payoff matrix and the update mechanisms. It is worth men-tioning that Press and Dyson have fundamentally and dramatically changed our understanding of this classic game byuncovering a new class of strategies, so-called "zero-determinant" strategies, which open a range of new possibilities for the study of coop-eration (Press and Dyson, 2012; Adami and Hintze, 2012; Hilbe et al., 2013).

In the traditional SPDG, there are two strategic types of indivi-duals: cooperator (*C*) and defector (*D*). If both players choose *C*, both get a payoff of magnitude *R*; if one defects while the other cooperates, *D* gets the game's biggest payoff, *T*, while *C* gets *S*; if both defects, both get *P*. This can be formalized in the form of a payoff matrix

$$
\begin{array}{cc}
 & C \quad D \\
C & R \quad S \\
D & T \quad P.
\end{array}
\tag{1}
$$

To be interesting, the game is characterized by the payoff ranking $T > R > P > S$, which guarantees that the Nash equilibrium of the game is mutual defection. For repeated games, the additional requirement $2R > T + S$ ensures that mutual cooperation is the globally best outcome. The "conventional value" $(T, R, P, S) = (b, 1, 0, 0)$ is chosen in this literature; the essentials of the game are pre-served in the sense that none of our findings and proofs are quan-titatively altered, if the parameter *P* is instead set as $P = \varepsilon < 0.001$ (so that $T > R > P > S$ is strictly satisfied). The parameter *b* ($b > 1$), which represents the advantage of defectors against cooperators, is there-fore the only parameter in the model.

In this work, we apply SPDG on a two-dimensional infinitely large regular lattice with Moore neighborhood, purely determi-nistic strategies, and no memories among players. Essentially, this simple and purely deterministic SPDG is an example of two-dimensional cellular automata (2D CAs) with two states and the neighborhood structure consisting of the two closest cells in each direction, for a total of 25 cells. Based on the heuristic analysis of frequency of cooperation, it is interesting to note that the mac-roscopic behaviors of this game are very sensitive to the initial conditions, which is the most practically significant property of chaos. According to the formal characterization of the notat-ion of chaos, chaos means deterministic behaviors that are very sensitive to the initial conditions. In other words, infinitesimal

perturbations of initial conditions for a chaotic dynamical system lead to large variations in behavior. Recently, chaotic phenomena of Chua's Bernoulli-shift rules with distinct parameters have been rigorously and thoroughly interpreted under the framework of symbolic dynamics (Jin et al., 2010; Chen et al., 2009; Chua and Pazienza, 2010a, 2010b). These analytical assertions shed light on the importance of gliders–a glider is a periodic structure moving into the evolution space, and essentially a subshift in view of symbolic dynamics (for formal definition see Jin et al., 2014; Jin and Chen, 2014; Cattaneo et al., 2000; Gianpiero and Luciano, 1998). The purpose of this paper is to provide the rigorous and firm explanations for three pivotal features of frequency of coop-eration via mathematical analysis, such as topological entropy and topologically mixing.

Findings discussed in the preceding references are based on computational experiments with lace doily, rose window and Persian carpet, as well as gliders (mobile localizations) in this game. This paper is not to report expressions or discoveries of gliders, but devoted to illustrating nonlinear explanations for the experimental simulations of frequency of cooperation via an ana-lytical method in the framework of symbolic dynamics, which demonstrates the underlying chaos of the extant gliders and their combinations in a subtle way. To enhance clarity and convenience, this work is concerned exclusively with three canonical gliders of this game with temptation-to-defect parameter $b = 1.66$, denoted by g_1–g_3, and their combinations. From the viewpoint of 2D symbolic dynamics, their chaotic dynamical properties are detailed analyzed, such as topological entropy and topologically mixing, and some quantitative explanations of their intrinsic complexity, the chaotically changing spatial patterns, and the indefinitely fluctuating proportion of cooperators are obtained.

2. Macroscopic properties

Basically, a cellular automata consists of the configuration space and local rules (Neumann and Burks, 1996; Wolfram, 2002). A 2D cellular automata is a two-dimensional array of identical automata (the cells), where each cell takes one of the values of 0 or 1. Each cell of the array is simultaneously updated to evolve a new state which is determined by the local rules. Consider SPDG played on the regular lattice with strategies *C* and *D*, denoted by 0 and 1, respectively, and periodic boundary conditions. Then, the agents in this game simultaneously play SPDG with their network neighbors consisting of the eight cells surrounding a central cell on a two-dimensional square lattice, and get payoffs according to the payoff matrix (1). The total payoff for each agent is the sum of all payoffs in these encounters with neighbors. Each player then mimic his neighbor's strategy with certainty by comparing his payoff in this step with his neighbors' payoffs. Therefore, the state of each cell at time *t* depends on 25 cells in its neighborhood at time $t - 1$, a total of 2^{25} possible patterns or local rules. As an illustration, one local rule is exhibited in Fig. 1.

The key quantity for characterizing the macroscopic behaviors of this system is the density of cooperators, f_c, which is widely adopted by averaging over several steps after the game reaches the equilibrium state. The conclusions of f_c reported in the previous literatures are always conducted as a function of *b* by averaging over some different random initial states. Note that a network with larger size will decrease the ensemble error. Consequently, the simulations are frequently performed with the population size of more than $N = 100 \times 100$. Here, the long-term time-asymptotic behaviors of f_c over discrete-time steps, denoted by $f_c(t)$, are explored with $b = 1.66$, and the network size of $N = 100 \times 100$ and $N = 200 \times 200$, as shown in Fig. 2. Samples of our experimental simulations are serially extracted from $f_c(t)$ after 10^6 generations,