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Sharing a resource with randomly arriving foragers

Pierre Bernhard[a,*], Frédéric Hamelin[b]

[a] BIOCORE team, INRIA-Sophia Antipolis Méditerranée, B.P. 93, F-06902 Sophia Antipolis Cedex, France
[b] AGROCAMPUS OUEST, UMR 1349 IGEPP, F-35042 Rennes Cedex, France

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ABSTRACT

We consider a problem of foraging where identical foragers, or predators, arrive as a stochastic Poisson process on the same patch of resource. We provide effective formulas for the expected resource intake of any of the agents, as a function of its rank, given their common functional response. We give a general theory, both in finite and infinite horizon, and show two examples of applications to harvesting a common under different assumptions about the resource dynamics and the functional response, and an example of application on a model that fits, among others, a problem of evolution of fungal plant parasites.

1. Introduction

The theory of foraging and predation has generally started with the investigation of the behavior of a lone forager [3,15] or of an infinite population of identical foragers, investigating the effect of direct competition [18,19,28] or their spatial distribution [7,16,17]. Then, authors investigated fixed finite groups of foragers in the concept of "group foraging" [4,8,9].

This article belongs to a fourth family where one considers foragers arriving as a random process. Therefore, there are a finite number of them at each time instant, but this number is varying with time (increasing), and a priori unbounded. We use a Poisson process as a model of random arrivals. Poisson processes have been commonly used in ecology as a model of encounters, either of a resource by individual foragers, or of other individuals [1,26]. However, our emphasis is on foragers (or predators) arriving on a given resource. There do not seem to be many examples of such setups in the existing literature. Some can be found, e.g. in [10–12], and also [29] (mainly devoted to wireless communications, but with motivations also in ecology).

In [11], the authors consider the effect of the possibility of arrival of a single other player at a random time on the optimal diet selection of a forager. In [10,12], the authors consider an a priori unbounded series of arrivals of identical foragers, focusing on the patch leaving strategy. In these articles, the intake rate as a function of the number of foragers —or functional response— is within a given family, depending on the density of resource left on the patch and on the number of foragers (and, in [12] on a scalar parameter summarizing the level of interference between the foragers). And because the focus is on patch leaving strategies, one only has to compare the current intake rate with an expected rate in the environment, averaged over the equiprobable ranks of arrival on future patches.

In the current article, we also consider an a priori unbounded series of random arrivals of identical foragers, but we focus on the expected harvest of each forager, as a function of its rank and arrival time. Our aim is to give practical means of computing them, either through closed formulas or through efficient numerical algorithms. These expressions may later be used in foraging theory, e.g. in the investigation of patch leaving strategies or of joining strategies [25].

In Section 2, we first propose a rather general theory where the intake rate is an arbitrary function of the state of the system. All foragers being considered identical, this state is completely described by the past sequence of arrivals and current time.

In Section 3, we offer three particular cases with specific resource depletion rates and functional responses, all in the case of "scramble competition" (see [10]). But there is no a priori obstruction to dealing also with interference. The limitation, as we shall see, is in the complexity of the dynamic equation we can deal with.

We only consider the case of a Poisson process of arrivals, making the harvesting process of any player a Piecewise Deterministic Markov Process (PDMP). Such processes have been investigated in the engineering literature, since [27] and [24] at least. As far as we know, the term PDMP (and even PDMDP for Piecewise

* Corresponding author.
E-mail addresses: Pierre.Bernhard@inria.fr (P. Bernhard), fhamelin@agrocampus-ouest.fr (F. Hamelin).

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Deterministic Markov Decision Process, but we have no decision here) was first introduced in [5]. Their control, the decision part, was further investigated, in e.g. [6,31] and a wealth of literature. Later articles such as [2,13] have concentrated on asymptotic properties of their optimal trajectories, and applications in manufacturing systems.

These articles (except [5] who proposes general tools for PDMP parallel to those available for diffusion processes) focus on existence and characterization of optimal control strategies. When they give means of calculating the resulting expected payoff, it is through a large (here infinite) set of coupled Hamilton–Jacobi (hyperbolic) partial differential equations. Here, we want to focus our attention on the problem of evaluating this payoff when the intake rates, the equivalent of strategy profiles of the control and games literature, for each number of players present on the common, are given; typically a known functional response. We take advantage of the very simple structure of the underlying jump process (discussed below), and of the continuous dynamics we have, to obtain closed form, or at least numerically efficient, expressions for the expected payoff, which we call *Value* for brevity.

2. General theory

2.1. Notation

Data: t_1, T, λ, $\{L_m(\cdot,\cdot)\,,\ m\in\mathbb{N}\}$.
Result sought $V_n(\cdot)$, $n\in\mathbb{N}$.

$t_1\in\mathbb{R}$	Beginning of the first forager's activity.
$T\in(t_1,\infty]$	Time horizon, either finite or infinite.
$t\in[t_1,T]$	Current time.
$m(t)\in\mathbb{N}$	Number of foragers present at time t.
t_m	Arrival time of the mth forager. (A Poisson process.)
τ_m	Sequence $(t_2,t_3,\dots t_m)$ of past arrival times.
$\mathcal{T}_m(t)\subset\mathbb{R}^{m-1}$	Set of consistent $\tau_{m(t)}$: $\{(\tau_m\vert t_1<t_2\cdots<t_m\le t\}$.
$\lambda\in\mathbb{R}_+$	Intensity of the Poisson process of arrivals.
$\delta\in\mathbb{R}_+$	Actualization factor (intensity of the random death process).
$L_m(\tau_m,t)\in\mathbb{R}_+$	Intake rate of all foragers when they are m on the common.
$M_m(t)\in\mathbb{R}_+$	Sum of all possible $L_{m(t)}$, for all possible $\tau_m\in\mathcal{T}_m(t)$.
$J_m(\tau_m)$	Reward of forager with arrival rank m, given the sequence τ_m of past arrival times. (A random variable.)
$V_1\in\mathbb{R}_+$	First forager's expected reward.
$V_m(\tau_m)\in\mathbb{R}_+$	Expected reward of the forager of rank m.
$J_m^{(n)}(\tau_m)$	Reward of player m *if the total number of arriving foragers is bounded by* n. (Random variable)
$V_m^{(n)}(\tau_m)\in\mathbb{R}_+$	Expectation of $J_m^{(n)}(\tau_m)$.

2.2. Statement of the problem

We aim to compute the expected harvest of foragers arriving at random, as a Poisson process, on a resource that they somehow have to share with the other foragers, both those already arrived and those that could possibly arrive later. At this stage, we want to let the process of resource depletion and foraging efficiency be arbitrary. We shall specify them in the examples of Section 3.

2.2.1. Basic notation

We assume that there is a single player at initial time t_1. Whether t_1 is fixed or random will be discussed shortly. At this stage, we let it be a parameter of the problem considered. Then identical players arrive as a Poisson process of intensity λ, player number m arriving at time t_m. The state of the system, (if t_1 is fixed) is entirely characterized by the current time t, the current number of foragers arrived $m(t)$, and the past sequence of arrival times that we call $\tau_{m(t)}$:

$$\forall m\ge 2\,,\qquad \tau_m:=(t_2,t_3,\dots,t_m),$$

a random vector. The intake rate of any forager at time t is therefore a function $L_{m(t)}(\tau_{m(t)},t)$.

Let the horizon be T, finite or infinite. We may just write the payoff of the first player as

$$J_1(t_1)=\int_{t_1}^{T}\mathrm{e}^{-\delta(t-t_1)}L_{m(t)}(t_2,\dots,t_{m(t)},t)\,\mathrm{d}t\,.$$

(We will often omit the index 1 and the argument t_1 of J_1 or V_1.) We shall also be interested in the payoff of the nth player arrived:

$$J_n(\tau_n)=\int_{t_n}^{T}\mathrm{e}^{-\delta(t-t_n)}L_{m(t)}(\tau_{m(t)},t)\,\mathrm{d}t\,.$$

(We shall often, in such formulas as above, write m for $m(t)$ when no ambiguity results.) The exponential actualization $\exp(-\delta t)$ will be discussed shortly. We always assume $\delta\ge 0$. In the finite horizon problem, it may, at will, be set to $\delta=0$.

2.2.2. Initial time t_1

In all our examples, the functions $L_m(\tau_m,\ t)$ only depend on time through differences $t-t_1$, or $t-t_m, t_m-t_{m-1},\dots t_2-t_1$. They are *shift invariant*. We believe that this will be the case of most applications one would think of. In such cases, the results are independent of t_1. Therefore, there is no point in making it random.

If, to the contrary, the time of the day, say, or the time of the year, enters into the intake rate, then it makes sense to consider t_1 as a random variable. One should then specify its law, may be exponential with the same coefficient λ, making it the first event of the Poisson process. In this case, our formulas actually depend on t_1, and the various payoff V_n should be taken as the expectations of these formulas.

One notationally un-natural way of achieving this is to keep the same formulas as below (in the finite horizon case), let $t_1=0$, and decide that, for all $m\ge 2$, t_m is the arrival time of the forager number $m-1$. A more natural way is to shift all indices by one, i.e. keep the same formulas, again with $t_1=0$, and decide that $\tau_m:=(t_1,t_2,\dots,t_m)$, and $\mathcal{T}_m(t)=\{\tau_m\mid 0<t_1<\dots<t_m\le t\}$.

2.2.3. Horizon T

The simplicity of the underlying Markov process in our Markov Piecewise Deterministic Process stems from the fact that we do not let foragers leave the resource before T once they have joined. The main reason for that is based upon standard results of foraging theory that predict that all foragers should leave simultaneously, when their common intake rate drops below a given threshold. (See [3,10,12].)

When considering the infinite horizon case, we shall systematically assume that the system is shift invariant, and, for simplicity, let $t_1=0$. A significant achievement of its investigation is in giving the conditions under which the criterion converges, i.e. how it behaves for a very long horizon. Central in that question is the exponential actualization factor. As is well known, it accounts for the case where the horizon is not actually infinite, but where termination will happen at an unknown time, a random horizon with an exponential law of coefficient δ. It has the nice feature to let a bounded revenue stream give a bounded pay-off. Without this discount factor, the integral cost might easily be undefined. In that respect, we just offer the following remark:

Proposition 1. *If there exists a sequence of positive numbers* $\{\ell_m\}$ *such that the infinite series* $\Sigma_m\ell_m$ *converges, and the sequence of functions* $\{L_m(\cdot)\}$ *satisfies a growth condition*

$$\forall m\in\mathbb{N}\,,\quad \forall\text{ sequences }(t_2,t_3,\dots,t_m,t)\,,\quad |L_m(t_2\dots,t_m,t)|\le\ell_m\,,$$