Mathematical Biosciences 266 (2015) 10–14

Contents lists available at ScienceDirect

Mathematical Biosciences

journal homepage: www.elsevier.com/locate/mbs

On the final size of epidemics in random environment

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ARTICLE INFO

Article history:
Received 15 November 2014
Revised 14 May 2015
Accepted 15 May 2015
Available online 23 May 2015

MSC:
92D30
60J27

Keywords:
Final epidemic size
SIR and SEIR model
Random environment
Basic reproduction number

ABSTRACT

This paper extends the final size result of the classical SIR epidemic model in constant and periodic environments to random environment. Conditionally on the basic reproduction number R_0 recently defined for random environment and the initial infected population fraction, we prove a final size result of an epidemic governed by the SIR model with time-depending parameters. The parameters are driven by an ergodic inhomogeneous time-periodic Markov process with finite state space. We also analyze the classical SEIR epidemic model in random environment.

1. Introduction

1.1. Model construction

Consider a closed population divided into three compartments: S for susceptible, I for infected and R for removed. We suppose that $S(t)$, $I(t)$ and $R(t)$ give the population proportion of each compartment at time t, so, $S(t) + I(t) + R(t) = 1$. This paper deals, in a wide part, with the well-known SIR epidemic model:

$$\begin{cases} \frac{d}{dt}S(t) = -a(t)S(t)I(t), \\ \frac{d}{dt}I(t) = a(t)S(t)I(t) - b(t)I(t), \\ \frac{d}{dt}R(t) = b(t)I(t), \end{cases} \tag{1}$$

with $a(t) = a_{\theta(t)}$ and $b(t) = b_{\theta(t)}$ be positive right continuous bounded functions describing the infection and recovery mechanisms, respectively. The environmental process θ is assumed to be an ergodic Markov process with state space $\{1, 2, \dots, K\}$ governed by an inhomogeneous T-periodic transition density $Q(t)$ (see [10], we note that in population dynamics $Q(t)$ is commonly used in place of $Q(t)^*$, where * stands for the transpose matrix operator, as is known in the probability theory) supposed to be irreducible and continuous with an initial distribution law μ. This choice expresses a natural mixture between periodic and random influences of environment. One important candidate external force that will influence parameters are the climatic conditions (see [12]) which seems for us that it has such evolution given by θ.

Following [5] (see also a known result for the periodic-coefficients equations [1]) one can check the existence of an unique T-periodic stationary probability $u(t) = (u_1(t), \dots, u_K(t))'$ on $\{1, 2, \dots, K\}$, where the symbol $'$ stands for the transpose operator, such that

$$\frac{d}{dt}u(t) = Q(t)u(t), \ \forall t > 0 \text{ and } u(0) = \mu. \tag{2}$$

We assume that (1) start by

$$S(0) = 1 - i < 1, \quad I(0) = i, \quad R(0) = 0. \tag{3}$$

Using sample path reasoning, one can show that the solution of (1) stays in $[0, 1]^3$, that S is decreasing, R is increasing and that $I_\infty := \lim_{t \longrightarrow \infty} I(t) = 0$.

1.2. The basic reproduction number

The classical concept of the basic reproduction number R_0 has been extended from constant environments in [9], to periodic environments in [7], and to random environments in [5]. One epidemiological ‘common’ interpretation of this quantity is “the number by which all infection rates should be divided to bring the epidemic to

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http://dx.doi.org/10.1016/j.mbs.2015.05.004
0025-5564/

the critical situation where neither exponential growth nor exponential decay occurs". In this part we give the mathematical definition of R_0 for (1). For this purpose, for any k in $\{1, 2, \ldots, K\}$ we define $\overline{u_k}$ to be the mean of $u_k(.)$; i.e.

$$\overline{u_k} := \frac{1}{T}\int_0^T u_k(s)\,ds. \tag{4}$$

For any vector $x = (x_1, \ldots, x_K)$, let

$$\bar{x} = \sum_{k=1}^{K} x_k \overline{u_k}, \tag{5}$$

be the mean of x weighted by the stationary probability u.

Using [13] (Example 2.2 (c), one can see that $u(t) = \mathcal{P}(t, 0)\mu$ where $\mathcal{P}(t, 0)$ is the transition matrix of θ and we can use the Lebesgue measure instead of the example-measure) we get an ergodic result; for all bounded functions f defined on $\{1, \ldots, K\}$, almost surely, we have

$$\lim_{t\longrightarrow\infty} \frac{1}{t}\int_0^t f(\theta(s))\,ds = \sum_{k=1}^{K} f(k)\overline{u_k}. \tag{6}$$

The basic reproduction number in our case is then given by $R_0 = \bar{a}/\bar{b}$. Indeed, let the second equation of (1) near the disease-free equilibrium (1, 0, 0), we have

$$\frac{d}{dt}I(t) = (a(t) - b(t))I(t). \tag{7}$$

Following [5], define the largest Lyapunov exponent of (7):

$$\begin{aligned}\lambda(a, b) &:= \lim_{t\longrightarrow\infty} \frac{1}{t}\log(I(s)) \\ &= \lim_{t\longrightarrow\infty} \frac{1}{t}\int_0^t (a(s) - b(s))ds.\end{aligned} \tag{8}$$

By (6), we have $\lambda(a, b) = \bar{\lambda} = \bar{a} - \bar{b}$. Then, the basic reproduction number is defined as the unique positive real number R_0 such that $\lambda(a/R_0, b) = 0$. Hence $R_0 = \bar{a}/\bar{b}$. The quantity $\bar{\lambda} = \bar{a} - \bar{b}$ is the average net infection rate, which defines the asymptotic growth rate of the linearized SIR model and we have the following simple result $\mathrm{sign}(\bar{\lambda}) = \mathrm{sign}(R_0 - 1)$.

1.3. The aim of the paper

Our goal is to study the final epidemic size R_∞, the limit of $R(t)$ as $t \longrightarrow \infty$, as a function of R_0 and i. We prove that : R_∞ is, almost surely, close to 0 if the basic reproduction number R_0 is strictly lower than 1 and i is close to 0, and is bigger than $1 - 1/R_0$, if R_0 is strictly bigger than 1 and $i \neq 0$.

The final epidemic size of the SIR model in constant environments; i.e. $(a(t), b(t)) = (a, b)$ for all t, is given implicitly by the formula

$$(1 - R_\infty)e^{R_0 R_\infty} = 1 - i, \tag{9}$$

where R_0 is equal to $\frac{a}{b}$. We must see this case as a first approximation of environmental variation, so a and b are the mean (in the sense of (5)) of some infection and recovery mechanisms list. Hence, R_∞ have the following asymptotic behavior (as function of i)

$$R_\infty \longrightarrow 0, \text{ on } \{R_0 < 1,\ i \simeq 0\}, \tag{10}$$

and

$$R_\infty \geq 1 - \frac{1}{R_0}, \text{ on } \{R_0 > 1,\ i \neq 0\}. \tag{11}$$

Arino et al. [2] study the final epidemic size in a constant environment for a general set of epidemic models. Recent work by Artalejo et al. [3] presents a continuous time Markov stochastic SIS and SIR epidemic models in a random environment, the authors give some results for quasi-stationary distribution and the distribution of the time to extinction. While in [11], a deterministic SIS model in a random environment is proposed. In [6], the authors prove that the same behavior of R_∞, in the sense of (10) and (11), happened in periodic environment for a suitable definition of R_0. Using similar techniques as in [6], this paper extends the latter results to the case of random environments.

The paper is divided into four parts. Section 2 gives an analytical proof of (10) and (11) in random environments, that is: if $R_0 < 1$, then for all $\varepsilon > 0$, there exists $\alpha > 0$ such that $i < \alpha$ implies $R_\infty < \varepsilon$ almost surely; if $R_0 > 1$ and $i \neq 0$, then $R_\infty \geq 1 - \frac{1}{R_0}$ almost surely. In section three we give a numerical example shows the validity of the results of the second section. In the fourth section, we investigate the final epidemic size in a two infection compartment model, typically, the SEIR model. The last section presents some conclusions. For details of the epidemiological significance of SIR (also the derivation of (9)) and SEIR system one can consult [12].

2. The SIR final size result

2.1. The sub-critical regime

Proposition 1. *If $R_0 < 1$, then for all $\varepsilon > 0$ there exists $\alpha > 0$ such that $R_\infty < \varepsilon$ a.s. for each $i < \alpha$.*

Proof. Because of $S(t) \leq 1$ for every time t, we have

$$\frac{d}{dt}I(t) \leq (a(t) - b(t))I(t), \tag{12}$$

so,

$$I(t) \leq i\exp\left[\int_0^t (a(s) - b(s))ds\right]. \tag{13}$$

For large time t, we have almost surely

$$\int_0^t (a(s) - b(s))ds \simeq \bar{\lambda}t. \tag{14}$$

One can conclude, almost surely, that $\int_0^\infty \exp[\int_0^t (a(s) - b(s))ds]$ is finite and so

$$C := \int_0^\infty b(t)\exp\left[\int_0^t (a(s) - b(s))ds\right]dt, \tag{15}$$

exists. We have $\frac{d}{dt}R(t) = b(t)I(t)$, then (almost surely)

$$R_\infty \leq iC. \tag{16}$$

The proposition then follows. □

2.2. The supercritical regime

Proposition 2. *If $R_0 > 1$ and $i \neq 0$, then $R_\infty \geq 1 - \frac{1}{R_0}$ almost surely.*

Proof. The proof goes on by contradiction. Suppose that

$$\Omega_0 := \left\{1 - R_\infty > \frac{1}{R_0}\right\} = \{R_0(1 - R_\infty) - 1 > 0\},$$

has a positive probability measure. We will work on Ω_0.

We know that

$$\frac{d}{dt}I(t) \geq [a(t)(1 - R_\infty) - b(t)]I(t) - a(t)I^2(t). \tag{17}$$

But $I_\infty = \lim_{t\longrightarrow\infty} I(t) = 0$. So, we can choose α satisfying $0 < \bar{a}\alpha < \bar{b}[(1 - R_\infty)R_0 - 1]$ and t_∞ such that $0 \leq I([t_\infty, \infty)) \leq \alpha$ and $I(t_\infty) > 0$.