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A multi-stage compartmental model for HIV-infected individuals: II – Application to insurance functions and health-care costs

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ABSTRACT

Stochastic population processes have received a lot of attention over the years. One approach focuses on compartmental modeling. Billard and Dayananda (2012) [1] developed one such multi-stage model for epidemic processes in which the possibility that individuals can die at any stage from non-disease related causes was also included. This extra feature is of particular interest to the insurance and health-care industries among others especially when the epidemic is HIV/AIDS. Rather than working with numbers of individuals in each stage, they obtained distributional results dealing with the waiting time any one individual spent in each stage given the initial stage. In this work, the impact of the HIV/AIDS epidemic on several functions relevant to these industries (such as adjustments to premiums) is investigated. Theoretical results are derived, followed by a numerical study.

1. Introduction

Over the century since Ross [2] first developed his mathematical model for a malaria process, there has been an extensive array of models introduced for numerous population processes, some being deterministic such as Ross' model, some being stochastic such as Bartlett's [3] early stochastic models for evolutionary processes followed by Bailey's [4] simple stochastic epidemic process. As a general rule (though obviously not true universally), equations governing deterministic models are 'easier' to solve than the more mathematically intractable equations governing stochastic processes. This is particularly so for stochastic models with nonlinear transition rates/probabilities. Yet, these nonlinearities are quite endemic to epidemic/disease processes. Our focus is with stochastic epidemic models; and in particular the impact of the HIV/AIDS epidemic on insurance and health-care functions.

Typically, many researchers have established the relevant set of differential-difference equations describing a given stochastic process (be these epidemic, queuing, biological, etc. models), and then explored techniques to solve these (largely intractable) equations. In more recent times, other researchers have set up their processes in a compartmental modeling framework. An extensive, but by no means exhaustive, review of some of the many approaches adopted over the years is in [1].

Billard and Dayananda [1] considered a compartmental model with transitions from one compartment (stage) to another and in addition allowed for death at any stage to occur. While their general approach is applicable to many population processes, their work was motivated by the need to establish guidelines for the insurance industry in particular (with parallel applications to the health-care industry, among others) as to the impact of the HIV/AIDS epidemic on insurance functions. Therefore, in that work, several probabilities and waiting time distributions for the compartmental process were derived. In this work, we derive insurance functions of interest. For example, insurance companies are vitally interested in future payouts (when and how much) for policies issued to their clients. A major concern relates to the impact that the HIV/AIDS epidemic has on these payments. These in turn are related to the premium levels clients are required to pay for continuous t-year life annuity and insurance policies (e.g.) and their long term effect ($t \to \infty$). Clearly, these entities are impacted by whether or not a potential policy-holder is healthy (i.e., is uninfected) or not (i.e., is already infected with HIV) at the time the policy is issued. Life expectancies and how, if at all, these are affected by the HIV/AIDS process are other functions of interest to insurers.

Accordingly, we give first, in Section 2, a description of the basic compartmental model, along with a summary of the approach used

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in Billard and Dayananda [1] to obtain the necessary distributions etc. Then, in Section 3, we derive insurance functions, against the backdrop that policy holders can become infected with HIV and be diagnosed with AIDS and how this diagnosis impacts the insurance entities. Section 4 looks briefly at how the ideas can be extended to health-care functions. The special case of exponential waiting times (usually not applicable for HIV/AIDS but can be valid for other diseases) is included, in Section 5. How these functions are affected by changing values of the model parameters is investigated in Section 6. This numerical study includes consideration of the impact of HIV/AIDS treatment to those known to have the disease.

2. Compartmental model – waiting time approach

In general, a compartmental process can be modeled as a multistage process with $X_j(t)$ individuals in stage $j = 0, \ldots, J$ at time $t > 0$. In the framework of epidemic processes, stage $j = 0$ corresponds to individuals who are initially susceptibles (i.e., uninfected by whatever disease is under study). Stages $j = 1, \ldots, m+1$ correspond to different levels of infection through which individuals successively pass. (Processes which allow a return to an earlier stage are accommodated by interpreting passage to a given stage as the last such visit to that stage.) Ultimately, individuals can die when they move into stage $j = m+2$. In the context of HIV/AIDS, the stage $j = 1$ is when initially infected with the HIV virus, and $j = m+1$ corresponds to diagnosis with AIDS. Transition rates, and/or probabilities, of moving from stage to stage are functions of the number of individuals in each stage; most such functions are nonlinear and so produce differential-difference equations that are mathematically intractable.

Instead of writing the model in terms of the variables $X_j(t)$, Billard and Dayananda [1] focused on waiting time distributions to move across stages. They also allowed for individuals to die from non-disease (such as non-HIV/AIDS) causes at any time; i.e., individuals could move directly to stage $j = m+2$. This feature is missing from most models even though it readily exists in many realistic situations (especially in insurance and health-care among other applications).

To summarize the Billard and Dayananda [1] (B&D) approach, and using their notation in the sense they defined it, let us define V_j to be the time an individual stays in stage j until moving out into stage $(j+1)$, $j = 0, \ldots, m+1$. Let us further define U_j, $j = 0, \ldots, m+1$, as the time an individual stays in stage j until death, i.e., until he moves directly into stage $m+2$. It is assumed that V_j and U_j are independent random variables. Suppose at time t an individual is in stage $S(t)$. A key entity (see Sections 3–5) is the probability $q_{ij}(t)$ that an individual is in stage j at time t given that he started in stage, i.e.,

$$q_{ij}(t) = P\{S(t) = j | S(0) = i\}, \quad i \leqslant j,\ i,j = 0, \ldots, m+2,\ t > 0. \tag{1}$$

From B&D (Theorem 1, Eq. (34)), we have, for $i \leqslant j,\ i,j = 0, \ldots, m+2$,

$$q_{ij}(t) = [P(Y_{ij} > t | W_k > 0, \quad k = i, \ldots, j-1) - P(Y_{i,j-1} > t | W_k > 0,$$
$$k = i, \ldots, j-1)] \times P(W_k > 0, \quad k = i, \ldots, j-1), \quad t > 0, \tag{2}$$

where

$$W_j = U_j - V_j, \tag{3}$$

$$Y_{ij} = \sum_{k=i}^{j} H_k \tag{4}$$

with

$$H_i = \min(U_i, V_i). \tag{5}$$

Here, when an individual survives stage j (without dying) to move into stage $(j+1)$, $W_j > 0$; but when an individual died while in stage $j, W_j < 0$. The variable H_j is the actual time an individual spends in stage j, by moving to the next stage after time V_j or by dying and moving to stage $m+2$ after time U_j, whichever happens first. Thence, the variable Y_{ij} in (4) is the total time an individual who started in stage i spends in stages $i, \ldots, j$ before moving into stage $j+1$.

We need expressions for the probabilities $P(W_k > 0,\ k = i, \ldots, j-1)$ and the conditional probabilities $P(Y_{ij} > t | W_k > 0,\ k = i, \ldots, j-1)$ of (2). Given its importance, these are derived in B&D (Sections 3.2 and 3.4, respectively) explicitly for the 4-stage model ($m = 1$). In those derivations, susceptibles were modeled as becoming infected with HIV at a Poisson rate with mean λ; hence the waiting time random variable V_0 is exponentially distributed with parameter λ. The waiting time distribution V_1 corresponds to the incubation period between infection with HIV and diagnosis with AIDS; thus a Weibull distribution with parameters $(\alpha, \beta = 2)$ was used. The waiting time distribution V_2 corresponding to the time of diagnosis to death from AIDS was assumed to follow an exponential distribution with parameter θ. The waiting times for death from non-AIDS causes, U_j, were assumed to be exponentially distributed random variables with parameters $\mu_j,\ j = 0, 1, 2$.

Thence, the survival probabilities $q_{ij}(t)$ can be obtained as in B&D (Section 3.5). The $q_{ij}(t)$ are summarized in Appendix A.1. As we shall see in Sections 3,4, these waiting time distributions are fundamental entities in the derivation of many insurance and health-care functions. While the derivations in Sections 3,4 focus on the four compartmental model ($m = 1$) with the numerical study of Section 6 including parameter values that easily accommodate the influence and impact of individuals undergoing HIV/AIDS treatment, an alternative model could include an additional stage (such as a "treatment" stage, so that $m = 2$). Another alternative model would be to adjust the current ($m = 1$) model's distributions appropriately so as to capture the effect of treatment. Thus, the model presented in the paper can be modified to accommodate the effects of drug therapies.

3. Insurance functions

There are numerous insurance functions of interest. Traditionally, insurance companies charge premiums, at a given percentage higher than the normal rates, to individuals or groups that are likely to be considered as having "higher risk" than normal. Since individuals exposed to HIV/AIDS are perceived to be in higher risk categories, insurers are particularly interested in those insurance functions relating to premium rates, expected payouts and life expectancies. Therefore, we limit our focus to but these illustrative few functions. The basic theory behind these entities can be found in any of a number of actuarial sources (e.g., Bowers et al. [5]).

3.1. Premiums for t-year pure endowment policy

Payouts and hence premium rates are typically expressed in terms of \$1 units. A \$1 (future) payout to an individual who has survived t units of time is currently worth \${$exp(-\delta t)$} where δ is the force of interest. This \$1 is paid only if the individual has survived, with that person being in stage $j,\ j \leqslant m+1$, at time t. If the present expected value for an individual currently in stage i is denoted by $\bar{E}_i(t)$, then

$$\bar{E}_i(t) = \sum_{j=i}^{m+1} (\text{present value of } \$1) q_{ij}(t).$$