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Consistent micro, macro and state-based population modelling

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ABSTRACT

A population system can be modelled using a micro model focusing on the individual entities, a macro model where the entities are aggregated into compartments, or a state-based model where each possible discrete state in which the system can exist is represented. However, the concepts, building blocks, procedural mechanisms and the time handling for these approaches are very different. For the results and conclusions from studies based on micro, macro and state-based models to be *consistent* (contradiction-free), a number of modelling issues must be understood and appropriate modelling procedures be applied. This paper presents a uniform approach to micro, macro and state-based population modelling so that these different types of models produce consistent results and conclusions. In particular, we demonstrate the procedures (*distribution*, *attribute* and *combinatorial expansions*) necessary to keep these three types of models consistent. We also show that the different time handling methods usually used in micro, macro and state-based models can be regarded as different integration methods that can be applied to any of these modelling categories. The result is free choice in selecting the modelling approach and the time handling method most appropriate for the study without distorting the results and conclusions.

1. Introduction

This paper focuses on *population models* – which are defined as models of dynamic systems with an integer number of discrete entities (individuals) such as plants, animals, patients, vehicles, molecules, atoms, data packages or entities of any kind. Such models are frequently used in ecology, epidemiology, demography and queuing systems, and are also important in physics, chemistry, biology, traffic planning, production and many other fields.

The crucial task in modelling is to preserve the characteristics of interest of the system under study. Four fundamental properties of a population system under study are of special interest for preservation in the model:

- The integer non-negative quality of the entities in the population.
- The continuous nature of time, which should at least be sufficiently well approximated in the model.
- The structural and temporal relations creating the dynamics of the system.
- Important irregularly occurring events of the system. These have to be characterised by an appropriate probabilistic representation in the model, because they cannot be described in detail.

This list could be extended with spatial aspects – but we restrict this paper to non-spatial models.

A system of interacting entities, a *population system*, can be modelled in three different but related ways: by a *micro approach* where each entity is separately described, by a *macro approach* where similar entities are lumped into compartments so only the number of entities in each compartment is recorded, or by a *state-based approach* where each discrete state in which the system can exist is explicitly represented and every transition between these states is specified by a conditional probability. The behaviour of micro and macro models is typically calculated by a numerical method (simulation), while state-based models, if sufficiently simple, can also be analysed analytically.

In various applications it is often found that e.g. a micro model produces results that are inconsistent with those from a macro model based on stochastic differential (or difference) equations. Furthermore, an analytic approach based on a state-based model may produce still other results. A number of studies compare the results from different approaches of modelling the same system and discuss the pros and cons of these approaches [1–6]. However, a vast number of studies in which the results and conclusions are dependent on the modelling approach lack such a discussion. Little has been done to explain how different approaches are related and what is needed for different approaches to produce consistent results.

The purpose of this paper is to provide a base for *consistent* (contradiction-free) micro, macro and state-based population

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modelling and simulation. Two models will here be said to be consistent if their outputs in terms of probability distribution/density functions are contradiction-free for relevant output quantities. This means that we allow for comparison between a micro model based on identifiable individuals that can be studied individually and macro or state-based models that only produce aggregated results. In particular, we will demonstrate the procedures (*distribution*, *attribute* and *combinatorial expansions*) necessary to keep these three types of models consistent. We will also show that the different time handling methods usually used in micro, macro and state-based models can be regarded as different integration methods that can be applied to any of these modelling categories.

An advantage of having consistency between micro, macro and state-based population modelling is that it allows for *an appropriate choice of type of model* instead of selecting it by routine. There are many aspects to this choice such as: nature of the system, purpose of the study, size of the model, execution time in computer simulation, possibility of including both discrete and continuous quantities, simplicity, transparency and communicability of the model, parameterisation and possibility of estimating parameters, validation, and possibility of simplifying the model without distorting the results and conclusions. Some of these issues are discussed later on. Furthermore, consistency between the three model types provides a powerful context where different approaches can contribute different types of insights to a study (see Fig. 1).

The possibility of combining theory and practice in a consistent way for micro and macro population modelling and simulation and having access to important results from the theory of stochastic processes are major advantages. In some cases deterministic models, embedded in the stochastic model, can be of value for e.g. mathematical analysis, model fitting, optimisation and sensitivity analysis – see [7].

To avoid making the presentation longer and more detailed than necessary, no distinction is made between the *continuous time* of the system under study and the *almost continuous time* using sufficiently small time-steps in the numerical model. Therefore, the notation $x(t)$ is usually used rather than x_t. So, for the sake of simplicity, this paper refers to exponential distributions even in the case of models with almost continuous time. To avoid the linguistic similarity between 'state variable' in a macro model and 'state' in a state-based model, the term *compartment* is in this paper used instead of state variable.

This paper is organised in the following way. In Section 2, micro, macro and state-based modelling are presented, while in Section 3 three possible time handling principles are introduced and consistency is discussed. In Section 4 we consider the merits and demerits of the three approaches. Finally, in Section 5 the findings are discussed in a broader perspective.

2. Micro, macro and state-based modelling

2.1. Introduction

A system under study is in general composed of so many pieces (e.g. atoms), has so many characteristics and is so complicated that it can never be modelled in all its details. The very essence of modelling is to build a parallel description, called the *model*, which is much simpler but otherwise preserves important characteristics and mechanisms of the system under study.

For population systems in general, the nature of the system under study includes discrete entities that interact irregularly in a dynamic context over a continuous time. This nature has to be preserved in the model unless it can be shown that further simplifications can perform the task in accordance with the overall aim.

Before scrutinising the different approaches it is necessary to use distinct terminology for the entities, their characteristics and changes over time. In the literature there are many synonyms and homonyms, but we use the following:

The system under study consists of *entities* of the same or different kinds. These entities may have a number of *characteristics* that are permanent or can be changed. The entities may interact with other entities and with the environment so that *changes* occur, or they may remain in a situation for a longer or shorter time.

A conceptual model is a *specification* of what to include or exclude from the system under study in accordance with the purpose of the study and practical considerations. This is a complete specification, but the conceptual model cannot be executed. The task is then to map the properties of the conceptual model into an executable micro, macro or state-based model.

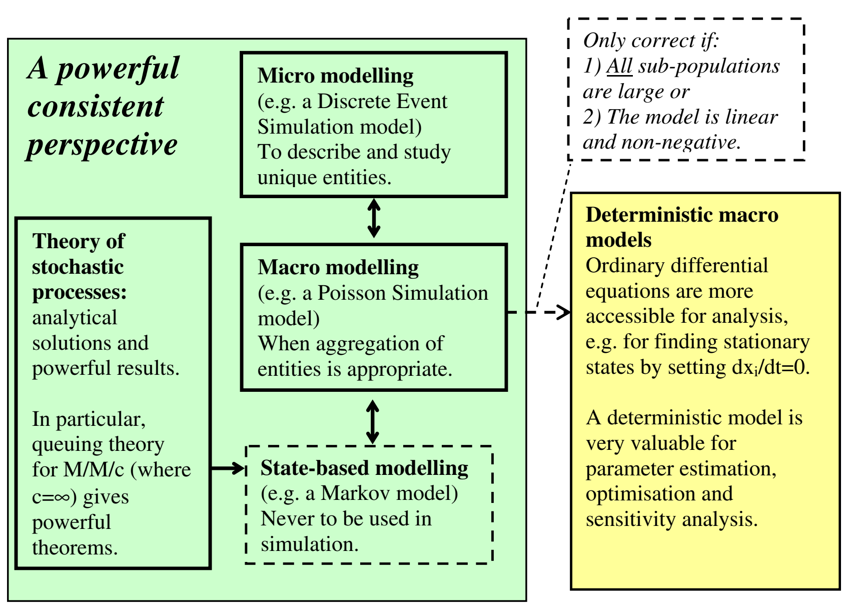

Fig. 1. Consistency brings a situation where theory and different simulation approaches are at hand for a powerful investigation of the system under study. It also shows when simplifications can be made. However, the simplification to a deterministic model is beyond the scope of this paper.