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Optimal harvesting and optimal vaccination

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Dedicated to the memory of Ovide Arino

Abstract

Two optimization problems are considered: Harvesting from a structured population with maximal gain subject to the condition of non-extinction, and vaccinating a population with prescribed reduction of the reproduction number of the disease at minimal costs. It is shown that these problems have a similar structure and can be treated by the same mathematical approach. The optimal solutions have a 'two-window' structure: Optimal harvesting and vaccination strategies or policies are concentrated on one or two preferred age classes. The results are first shown for a linear age structure problem and for an epidemic situation at the uninfected state (minimize costs for a given reduction of the reproduction number) and then extended to populations structured by size, to harvesting at Gurtin–MacCamy equilibria and to vaccination at infected equilibria.

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Keywords: Structured population; Harvesting; Culling; Infectious disease; Vaccination; Optimization

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1. Introduction

Vaccination of a population against the spread of an infectious disease and harvesting from a population have much in common. Harvesting removes individuals from a population with the result that population growth is slowed down. The gain from the harvest is increasing with the number of culled individuals. The possible gain and the resulting population loss must be balanced. Thus harvesting leads to several optimization problems. A vaccination campaign removes individuals from the susceptible part of the population in order to decrease the reproduction number of the disease or the level of prevalence. Again, the expenses incurred by the vaccination program must be balanced against the possible benefits which leads to the question of an optimal vaccination policy.

Harvesting models for structured populations are based on Leslie type matrix models [9,13], on stage models in continuous time [14], or on the Sharpe-Lotka–McKendrick model and the Gurtin–MacCamy model [4] which has the form of a partial differential equation with boundary condition for the time dependent population distribution $u(t, a)$. Harvesting is introduced as an additional death term $\psi(t, a)$. There is a great variety in the formulation of the problem. The problem can be time-dependent or stationary, the harvested amount $\psi(t, a)$ can be absolute or proportional to population size. A distinctive feature is whether the harvesting process is connected to population control or not. Beddington and Taylor [3] introduced the concept of maximum sustainable yield (MSY), i.e., maximizing yield without driving the population to extinction and perhaps respecting further ecological or social side conditions. Rorres and Fair [32,33] consider harvesting absolute amounts, Gurtin and Murphy [17] study a non-linear problem where natural mortality depends on population size and the harvesting term is $\psi(t, a) = E(t)u(t, a)$, thus harvesting is independent of age, see also [31]. The goal is to maximize yield for a given initial data and a prescribed final time and population level. Gurtin and Murphy [18,27], Murphy and Smith [28] consider similar but more complex problems. Brokate [4] studies a (non-linear) Gurtin–MacCamy system with harvesting over a finite time interval with uniformly bounded harvesting effort from a given initial distribution. In [2] a periodic harvesting problem is studied which in some sense appears more general than the approach of the present paper. However, in [2] a positive immigration rate plays a crucial role.

The variety of different vaccination models is still greater, reflecting various features of common diseases [1]. Standard vaccination models are based on the age structured Kermack–McKendrick SIR model. Vaccination is introduced as a term that removes individuals from the class of susceptible individuals [8,10]. Again one can consider time-dependent problems where certain goals must be achieved within a finite time horizon with minimal effort, say, or stationary models that describe optimal policies at equilibrium. Cairns [5] studies the effect of vaccination policies on the Jacobian at the uninfected equilibrium in multigroup epidemic models. Greenhalgh [15] considers vaccination of fixed proportions of the population at given ages to stabilize the uninfected equilibrium. Rouderfer and Becker [34] take loss of immunity into account. Vaccination policies in multigroup situations can lead to backward bifurcations, see [23,19,20]. The comparison of optimization concepts is of special importance since different optimization concepts do lead to different solutions. This statement sounds trivial, but it bears major problems when it comes to the point to draw practical conclusions [11,24].