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The entropy of the life table: A reappraisal

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The life table entropy provides useful information for understanding improvements in mortality and survival in a population. In this paper we take a closer look at the life table entropy and use advanced mathematical methods to provide additional insights for understanding how it relates to changes in mortality and survival. By studying the entropy (H) as a functional, we show that changes in the entropy depend on both the relative change in life expectancy lost due to death ($e^{\dagger}$) and in life expectancy at birth (e_0). We also show that changes in the entropy can be further linked to improvements in premature and older deaths. We illustrate our methods with empirical data from Latin American countries, which suggests that at high mortality levels declines in H (which are associated with survival increases) linked with larger improvements in e_0, whereas at low mortality levels $e^{\dagger}$ made larger contributions to H. We additionally show that among countries with low mortality level, contributions of $e^{\dagger}$ to changes in the life table entropy resulted from averting early deaths. These findings indicate that future increases in overall survival in low mortality countries will likely result from improvements in $e^{\dagger}$.

1. Introduction

The life table is perhaps the most useful tool in mortality analyses, as it summarizes the mortality experience of a population at a given point in time into a set of simple indicators (Preston et al., 2000). For example, life expectancy, a by-product of the life table, has been used extensively and widely as a measure of population health in national and international contexts (United Nations, 2012). Other life table measures such as the life table entropy, however, have received much less attention, although the entropy could also be considered an equally useful indicator for understanding improvements in mortality and survival in a population (Wilmoth and Horiuchi, 1999).

In this paper we take a closer look at the life table entropy and provide additional insights for understanding how it relates to changes in mortality and survival. Unlike previous work that relied on univariate calculus (e.g., Demetrius, 1974, 1975, 1976, 1978, 1979; Goldman and Lord, 1986; Keyfitz, 1977), we provide a more rigorous development and a further description of the life table entropy using the calculus of variations. This approach has previously been used in demographic research (Arthur, 1984; Beltrán-Sánchez and Soneji, 2011; Preston, 1982), and as we show, it provides us with additional tools to deepen our understanding of the population entropy and overall population survival. We focus, in particular, on a widely used measure of mortality improvement – life expectancy at birth (which represents the average length of life in the survival curve of a population) – and an additional measure called $e^{\dagger}$ that has recently been proposed as a marker of lifespan inequality (Zhang and Vaupel, 2009). For example, averting deaths at younger ages (premature deaths) is associated with reductions in lifespan inequality (Zhang and Vaupel, 2008). Recent evidence from 40 countries shows a negative correlation between life expectancy (e_0) and lifespan disparity ($e^{\dagger}$) from 1840 to 2009, with most of the increase in life expectancy resulting from improvements in premature deaths (Vaupel et al., 2011). The authors conclude that improvements in life expectancy at birth can also be accompanied by reductions in lifespan disparity ($e^{\dagger}$). In this paper we provide a mathematical foundation for these empirical findings by linking changes in the life table entropy, life expectancy at birth, and lifespan disparity. We demonstrate, mathematically and empirically, that changes in the entropy depend on both the relative change in life expectancy lost due to death ($e^{\dagger}$) and in life expectancy at birth (e_0). We also show that changes in the entropy can be further linked to averting premature and older deaths. These results provide important implications for understanding current and future changes in the overall survival of a population. For instance, using data from Latin American countries for 1950–2005, we show that at low mortality levels changes in $e^{\dagger}$ contributed the

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most to overall survival, indexed by the entropy, which resulted from improvements in premature deaths. This implies that in these countries future increases in overall survival will likely come from changes in $e^\dagger$ and that these improvements are likely to reduce lifespan inequality as a result of averting early deaths (Zhang and Vaupel, 2008, 2009).

The paper is organized as follows. We begin in Section 2 with a brief overview of the mathematical definitions of the mortality and survival functions, and the life expectancy and entropy (for the interested reader, Appendix A.1 contains a brief literature review of the entropy). We then review how the entropy is used to measure relative changes in life expectancy in Section 2.1, and discuss the functional nature of the entropy in Section 2.2. We present our main results in Sections 2.3–2.4, where we use the calculus of variations (reviewed in Appendix B) to show that changes in the entropy depend on both the relative change in life expectancy lost due to death ($e^\dagger$) and in life expectancy at birth (e_0) – c.f. (2.5) – and also provide a new way to describe the effect of changes in the mortality function on the population entropy (c.f. Proposition 2). In Section 3 we further link changes in the entropy with improvements in premature and older deaths in relation to e_0 and $e^\dagger$. Section 4 applies our results to mortality data from 18 Latin American countries from about 1950 to 2008. Therein we discuss our finding that at high mortality levels declines in H (which are associated with survival increases) linked with larger improvements in e_0, whereas at low mortality levels $e^\dagger$ made larger contributions to H. We end with concluding remarks in Section 5.

2. The entropy

The life table entropy is commonly used throughout demography to study the relative changes in life expectancy associated with changes in age-specific mortality rates. In this section we review the construction of the entropy due to Keyfitz (1977) (see Appendix A.1 for a brief history), and then present our main analytical results.

2.1. The demographic motivation for introducing the entropy

Let $\mu(x)$ be the force of mortality at age x. The probability of surviving from birth to age x is then

$$S(x) = e^{-\int_0^x \mu(s)\,ds}, \tag{2.1}$$

so that *life expectancy* at age x is given by

$$e(x) = \int_x^\infty e^{-\int_0^a \mu(s)\,ds}\,da.$$

In many of the situations of interest to us in this paper, x is fixed and $\mu(s)$ may vary. For instance, we may be interested in studying changes in life expectancy at birth (which implies that $x = 0$). We therefore introduce the following notation to reflect these cases:

$$S_x[\mu(s)] = e^{-\int_0^x \mu(s)\,ds}, \qquad e_x[\mu(s)] = \int_x^\infty e^{-\int_0^a \mu(s)\,ds}\,da. \tag{2.2}$$

Consider now a relative increase $\epsilon > 0$ in μ – that is, a proportional increase in μ at all ages – similar to that proposed by Keyfitz (1977). Then the new mortality function is $(1+\epsilon)\mu(s)$ (note that $\Delta\mu = \epsilon\mu$, so that $\Delta\mu/\mu = \epsilon$), the new probability of surviving from birth to age x is

$$S_x[(1+\epsilon)\mu(s)] = e^{-\int_0^x (1+\epsilon)\mu(s)\,ds} = \left(e^{-\int_0^x \mu(s)\,ds}\right)^{1+\epsilon} = \left(S_x[\mu(s)]\right)^{1+\epsilon},$$

and the new life expectancy at age x is

$$e_x[(1+\epsilon)\mu(s)] = \int_x^\infty S(a)^{1+\epsilon}\,da.$$

Without loss of generality, let us specialize to the most studied case of life expectancy—life expectancy at birth:

$$e_0[(1+\epsilon)\mu(s)] = \int_0^\infty S(a)^{1+\epsilon}\,da.$$

We expect the relative increase in mortality to cause a relative decrease in life expectancy. To measure this decrease, Keyfitz and Caswell (2005, sec. 4.3.1) calculate $de_0/d\epsilon|_{\epsilon=0}$ and then consider ϵ to be finite but small to arrive at the approximation

$$\frac{\Delta e_0}{e_0} \approx \left(\frac{\int_0^\infty S(x)\ln\left(S(x)\right)\,dx}{\int_0^\infty S(x)\,dx}\right)\epsilon. \tag{2.3}$$

Since $0 \le S(x) \le 1$ (this follows from (2.1)), the ratio in the parentheses is negative, confirming our expectation that a relative increase in mortality should result in a relative decrease in life expectancy. Accordingly, the negative of the expression in parentheses is known as the *entropy* of the life table, and is customarily denoted by H. More formally, we make the following definition.

Definition 1. Given a survival function $S(x)$, the quantity defined by

$$H[S(x)] = -\frac{\int_0^\infty S(x)\ln(S(x))\,dx}{\int_0^\infty S(x)\,dx} \tag{2.4}$$

is called the **entropy of the population**.

We will explain the bracket notation in the next section, but for now let us note that the approximation in (2.3) suggests the following interpretation for H (Goldman and Lord, 1986): a small proportional increase ϵ in the death rate at all ages results in a proportional decrease in life expectancy of approximately H times ϵ. For example, for $H = 1$ "when the death rates at all ages increase by 1 percent, the expectation of life diminishes by 1 percent Keyfitz and Caswell (2005, Sec. 4.3.1)". Thus, H measures how relative changes in the mortality function affect the relative change in life expectancy of a population. In other contexts H has other interpretations (see Appendix A.2), but it is commonly known to be "in general highly sensitive to variations in age-specific mortality" Demetrius (1979) (Appendix A.3 contains a more thorough discussion of this point), which makes it a useful tool for characterizing a population's survivorship.

2.2. Understanding the life table entropy (H) as a functional of the survival function (S) and the force of mortality (μ)

The preceding analysis described the effect on H of a specific change in the mortality function $\mu(x)$ (and consequently, by (2.2), in $S(x)$). This suggests that we view H as a *functional*—a quantity whose input is a function and whose output is a real number. Indeed, as (2.4) makes clear, H is a functional of $S(x)$, since it takes as input a survival function $S(x)$ and outputs a real number (this is why we have used the $H[S(x)]$ notation). Similarly, H can also be seen as a functional of $\mu(x)$, in which case we write $H[\mu(x)]$.

Functionals are similar to functions, except that the "independent variable" is now a function. To better see this important distinction (and also the functional nature of H), consider the so-called *hyperbolic mortality* example, where

$$\mu(x) = \frac{a}{s_0 - x}, \qquad S(x) = \left(1 - \frac{x}{s_0}\right)^a,$$

$$H[\mu(x)] = H[S(x)] = \frac{a}{a+1}.$$