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Evolutionary game dynamics with non-uniform interaction rates

Christine Taylor*, Martin A. Nowak

Program for Evolutionary Dynamics, Department of Mathematics, Department of Organismic and Evolutionary Biology, Harvard University, Cambridge, MA 02138, USA

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Abstract

The classical setting of evolutionary game theory, the replicator equation, assumes uniform interaction rates. The rate at which individuals meet and interact is independent of their strategies. Here we extend this framework by allowing the interaction rates to depend on the strategies. This extension leads to non-linear fitness functions. We show that a strict Nash equilibrium remains uninvadable for non-uniform interaction rates, but the conditions for evolutionary stability need to be modified. We analyze all games between two strategies. If the two strategies coexist or exclude each other, then the evolutionary dynamics do not change qualitatively, only the location of the equilibrium point changes. If, however, one strategy dominates the other in the classical setting, then the introduction of non-uniform interaction rates can lead to a pair of interior equilibria. For the Prisoner's Dilemma, non-uniform interaction rates allow the coexistence between cooperators and defectors. For the snowdrift game, non-uniform interaction rates change the equilibrium frequency of cooperators.

Keywords: Evolutionary game dynamics; Replicator dynamics; Non-uniform interaction rates; Prisoner's Dilemma; Snowdrift game; Kin selection; Evolutionary stability

1. Introduction: replicator equation with uniform interaction rates

Consider a two strategy game with payoff matrix

	A	B
A	a	b
B	c	d

Strategy A receives payoffs a and b when playing against strategy A and B, respectively. Strategy B receives payoffs c and d when playing against A and B, respectively. We denote by x and y the frequency of individuals adopting strategy A and B, respectively. We have $x+y=1$.

With uniform interaction rates, where players interact with each other indiscriminantly, the selection dynamics can be described by the standard replicator equation (Taylor and Jonker, 1978; Hofbauer et al., 1979; Hofbauer and Sigmund, 1998, 2003):

$$\dot{x} = x(f_A - \phi)$$
$$\dot{y} = y(f_B - \phi). \quad (1)$$

The fitness of A and B players are linear functions of x, given by

$$f_A = ax + by,$$
$$f_B = cx + dy.$$

The average fitness of the population is given by

$$\phi = f_A x + f_B y.$$

The replicator equation assumes that the rate (or probability) of interaction between two players is independent of their strategies.

There are three generic evolutionary outcomes:

(1) A dominates B: if $a>c$ and $b>d$, then the entire population will eventually consist of A players. The only stable equilibrium is $x=1$. A is a strict Nash equilibrium, and therefore an evolutionarily stable

*Corresponding author. Fax: +1 617 496 4629.
E-mail address: taylor4@fas.harvard.edu (C. Taylor).

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strategy (ESS), while B is not. We use the notation $A \longleftarrow B$.

(2) A and B co-exist in a stable equilibrium: if $a<c$ and $b>d$, then the interior equilibrium $x=(b-d)/(b+c-a-d)$ is stable. Neither A nor B are Nash equilibria. We use the notation $A \rightarrow \leftarrow B$.

(3) A and B are bi-stable: if $a>c$ and $b<d$, the interior equilibrium $x=(d-b)/(a+d-b-c)$ is unstable. The two boundary points, $x=0$ and 1 are attracting. A and B are both strict Nash equilibria. We use the notation $A \longleftrightarrow B$. If $a+b>c+d$, then strategy A is risk dominant. It has the larger basin of attraction. If $a>d$, then strategy A is Pareto optimal, i.e. there is no other strategies the two players can employ to have payoff at least as high as a, and at least one player having payoff higher than a.

2. Replicator equation with non-uniform interaction rates

Now suppose that the probability of interaction between two players is not independent of their strategies. Analogous to a chemical reaction, an A player interacts with another A player with reaction rate r_1, an A player and a B player interact with reaction rate r_2, and a B player interacts with another B player with reaction rate r_3.

$$A + A \xrightarrow{r_1} AA,$$
$$A + B \xrightarrow{r_2} AB,$$
$$B + B \xrightarrow{r_3} BB.$$

We assume that the fitness of individuals is determined by the average payoff over a large number of interactions. Therefore, the fitness of A and B players are non-linear functions of x and y, given by

$$f_A = \frac{ar_1x + br_2y}{r_1x + r_2y},$$
$$f_B = \frac{cr_2x + dr_3y}{r_2x + r_3y}.$$

The rates r_1, r_2 and r_3 are non-negative. The normal replicator equation with uniform interaction rates corresponds to the special case $r_1 = r_2 = r_3 > 0$. If $r_1 > r_2$ and $r_3 > r_2$, then players prefer to interact with their own kind. If, however, $r_1 < r_2$ and $r_3 < r_2$, then mixed interactions (between A and B) are more likely.

2.1. A comparison with kin selection

In the context of kin selection (Hamilton, 1964), games between relatives have been studied in the late 1970s by Grafen (1979), Hines and Maynard Smith (1979), Mirmirani and Oster (1978), Orlove (1978, 1979a,b), and Treisman (1977). The "inclusive fitness" approach simply modifies the original payoff matrix M to

	A	B
A	$(1+r)a$	$b+rc$
B	$c+rb$	$(1+r)d$

where r is the coefficient of relatedness. In this framework, all individuals of the population are assumed to be related equally. The payoff to a player in a pairwise interaction is the sum of his own payoff plus r times the opponent's payoff. Grafen proposed a "personal fitness" approach to account for the fact that an individual is more likely to play an opponent with the same strategy. In Grafen's model,

$$f_A = ra + (1-r)(ax+by), \quad f_B = rd + (1-r)(cx+dy),$$

where r is the probability than a player will meet an opponent with the same strategy because of some genetic or social relationship.

More recently, (Tao and Lessard, 2002) developed the ESS theory for frequency-dependent selection in family-structure populations. In particular, they use the payoff matrix

	A	B
A	$(1+r/2)a$	$b+cr/2$
B	$c+br/2$	$(1+r/2)d$

where r is the probability to interact with a sib, to show the effect of kin selection involving full sibs on ESS conditions.

All these approaches contain linear fitness functions. In contrast, our model is based on non-linear fitness functions and can therefore not be studied by a simple transformation of the payoff matrix. In our model, there will be new dynamical features which are not present in the standard replicator equation.

Queller (1985) used a definition of relatedness which depends on the covariance of the strategies of interacting individuals. Non-uniform interaction rates lead to high covariance of players' strategies, and hence a degree of relatedness in Queller's formulation.

2.2. Invariant transformations

For the standard replicator equation, there exist some useful transformations of the payoff matrix that do not change the evolutionary dynamics. Here we show that some of these transformations can also be used for non-uniform interaction rates.

Consider the equations

$$\dot{x} = x(f_A - \phi),$$
$$\dot{y} = y(f_B - \phi),$$

with $\phi = xf_A + yf_B$ and

$$f_A = \frac{ar_1x + br_2y}{r_1x + r_2y} \quad \text{and} \quad f_B = \frac{cr_2x + dr_3y}{r_2x + r_3y}.$$