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A comprehensive explanation and exercise of the source terms in hyperbolic systems using Roe type solutions. Application to the 1D-2D shallow water equations

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ABSTRACT

Powerful numerical methods have to consider the presence of source terms of different nature, that intensely compete among them and may lead to strong spatiotemporal variations in the flow. When applied to shallow flows, numerical preservation of quiescent equilibrium, also known as the well-balanced property, is still nowadays the keystone for the formulation of novel numerical schemes. But this condition turns completely insufficient when applied to problems of practical interest. Energy balanced methods (E-schemes) can overcome all type of situations in shallow flows, not only under arbitrary geometries, but also with independence of the rheological shear stress model selected. They must be able to handle correctly transient problems including modeling of starting and stopping flow conditions in debris flow and other flows with a non-Newtonian rheological behavior. The numerical solver presented here satisfies these properties and is based on an approximate solution defined in a previous work. Given the relevant capabilities of this weak solution, it is fully theoretically derived here for a general set of equations. This useful step allows providing for the first time an E-scheme, where the set of source terms is fully exercised under any flow condition involving high slopes and arbitrary shear stress. With the proposed solver, a Roe type first order scheme in time and space, positivity conditions are explored under a general framework and numerical simulations can be accurately performed recovering an appropriate selection of the time step, allowed by a detailed analysis of the approximate solver. The use of case-dependent threshold values is unnecessary and exact mass conservation is preserved.

1. Introduction

There are a unaffordable number of processes over the earth surface where a common agent is present: water. Water can participate in different ways: as a result of a high porosity in landslide events or as almost pure water in rivers. Many geophysical or environmental flows in earth have another relevant characteristic: the geometrical scales presented in the problem allow us to define them mathematically as shallow type flows. The vertical scales can be considered very small if compared with the horizontal ones. This aspect ratio appears in channels, rivers, oceans or even the atmosphere, but also in debris flows, landslides and tsunamis. All these processes can be mathematically modeled and are defined as flows of hyperbolic nature. Their importance makes necessary the development of predictive tools. Predictive tools were first derived for gas dynamics, and the results were applied next to the shallow water equations. Shallow type flows are hyperbolic but not strictly hyperbolic. Their characteristics require the development of novel numerical techniques. Among them, most advanced numerical predictive methods consider partial results, e.g. the well-balanced property. This property is a particular case of energy-balanced or E-schemes that have a significant advantage: they provide accurate results when using a small amount of information. This amount of information can be measured as the number of computational cells where data is stored. When the number of cells decreases, the computational cost also does, allowing the integration of more processes in time and space. A complete understanding of solvers accounting for the presence of source terms also allows to successfully predict the behavior of sophisticated terms when applied to cases of not pure water floods, such as mud/debris floods, where unsteady flow phenomena includes stop and go mechanisms. In this way, it is possible to analyze the relative importance of the shear stresses versus bottom topography variations, allowing a correct tracking of the fluid moving boundaries.

Realistic applications of conservation laws involve the presence of source terms dominating the solution, where the flux gradients are nonzero but exactly balanced by source terms in steady

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situations (LeVeque, 1998). When trying to reproduce numerical solutions with discontinuities in both the conserved variables and the source terms, the mathematical formulation of the governing equations and the selection of the numerical scheme is of utmost importance. Fractional step methods have been widely used to involve the presence of source terms in the solution (LeVeque, 2002), but from their earliest developments, well balanced numerical schemes (Bermudez and Vázquez-Cendón, 1994; Greenberg and LeRoux, 1996; LeVeque, 1998) have gained maturity to progressively become methods of choice for the numerical simulation of conservation laws with source terms. When applied to the shallow water equations (SWE), the preservation of motionless steady state or quiescent equilibrium over irregular geometries has been the keystone for the construction of numerical schemes in the context of shallow flows (Audusse et al., 2004; García-Navarro and Vázquez-Cendón, 2000; Hubbard and García-Navarro, 2000; Liang and Borthwick, 2009; Zhou et al., 2001).

Riemann solvers derived for the homogeneous case, in combination with a suitable treatment of the source terms, are able to ensure quiescent equilibrium when solving the SWE. This property can be ensured by expressing the equations following a deviatoric formulation (Bollermann et al., 2013; Liang and Borthwick, 2009; Zhou et al., 2001), replacing the water depth by the water surface elevation as conserved variable. But changes in the selection of the variables have consequences. When selecting the deviatoric formulation, the approximate solution is single valued and even though in presence of bed discontinuities level surface is constant in cases of quiescent equilibrium, this solution is no longer valid in general problems over a bottom step, invalidating the use of classical Riemann solvers (Harten et al., 1983; Roe, 1981; Toro et al., 1994). The most noticeable consequence is the inability of such type of solvers to ensure an exact preservation of the mass discharge in steady cases for any type of flow regime (Murillo and García-Navarro, 2012a).

Also, as pointed out by Xing and Shu (2014) when solving the SWE two types of difficulties are often encountered: the preservation of steady state solutions and the preservation of water height positivity. The SWE admit the general moving water equilibrium and require exactly well-balanced methods in cases with moving water equilibrium (Díaz et al., 2013; Fjordholm et al., 2011; Noelle et al., 2007; Xing, 2014). The well balanced numerical property in cases of quiescent equilibrium is a particular case. In Murillo and García-Navarro (2012b); 2014) and Navas-Montilla and Murillo (2015); 2016) exactly well-balanced methods, named energy balanced numerical schemes and hereafter referred as E-schemes, able to reproduce exactly steady solutions with independence of the mesh refinement, were presented. On the other hand, non-physical negative water height becomes problematic when computing simulations as the eigenvalues do not determine the time step size as a result of the not pure hyperbolic characteristic of the system of equations (Xing and Shu, 2014; Xing and Zhang, 2013; Xing et al., 2010). The use of case-dependent threshold values for the water depth, limiting the computational domain in the computation of the flow advance over dry bed, is the current tendency to avoid numerical difficulties (Delis et al., 2011a). Other techniques involve velocity based limiters (Delis et al., 2011b; Vater et al., 2015) to control the stability at wet/dry fronts.

Riemann Problems (RP) in not strictly hyperbolic system of equations involve complex exact solutions, as the presence of the source terms may lead to resonant problems. These resonant problems include cases where characteristic speeds may coincide or cases where the total number of waves involved in the solution are larger than the number of characteristic fields (LeFloch and Thanh, 2011). An extensive review of approximate solutions to discontinuous problems in nonlinear hyperbolic systems can be found in LeFloch and Mishra (2014). Convergence to the exact solution can be ensured using appropriate Augmented solvers (George, 2008; LeVeque, 1998; Murillo and García-Navarro, 2010b; 2012a). Augmented solvers provide suitable explanations to the influence of the source terms in the numerical solution. They include an extra wave associated to the presence of the source terms in the approximate solution. The aforementioned possible computation of non-physical negative water height was explained and remedied in Murillo and García-Navarro (2010b); 2012a) by the description of the internal structure of the associate approximate Riemann solution.

Even though a great variety of works that focus on the preservation of height positivity by exploring the effects of bed slope terms can be found in literature, when moving to realistic applications, the discretization of frictional source terms is essential to provide accurate results, independently of the friction stress model chosen. When the source term discretization of shear stress is not considered in the context of the approximate solution used, it can not only spoil the solution accuracy, but the numerical computation may become unstable and fail (Burguete et al., 2008; Murillo et al., 2006). Fractional explicit step methods lead to oversized discrete friction forces that ruin the simulation, and although an implicit treatment of the resistance source term ensures stability, an exact balance among fluxes and source terms is not generated (Murillo et al., 2009), leading to undesirable non uniform discharge values. That is, convergence to the exact solution can never be provided. The upwind unified treatment of boundary shear stress ensures exact conservation of discharge in steady cases, but being an explicit treatment, the appearance of non-physical negative values of water depth and the selection of the time step size, become again problematic (Burguete et al., 2008; Murillo et al., 2008), as in those numerical schemes where bed slope effects are only analyzed (Xing and Zhang, 2013).

The definition of appropriate numerical schemes involving bed variations can be envisaged using families of paths (Parés, 2006) connecting the left and right states of the RP (Maso et al., 1995). But even in cases where only discontinuous bed level is considered, the selection of families of paths is not a trivial task (Castro et al., 2008; Hou and LeFloch, 1994; LeFloch and Thanh, 2011). One commonly used strategy is based on supplementing the initial set of equations shaping the SWE with another extra equation, expressing a nil time derivative of the bed level surface (Díaz et al., 2013; Rosatti and Begnudelli, 2010). The results show that although Godunov-type path-consistent schemes do converge with mesh refinement, they do not necessarily converge to the physically relevant or correct solution (LeFloch and Mishra, 2014). Families of paths cannot be generalized when highly nonlinear relations appear (Murillo and García-Navarro, 2010a; Rosatti et al., 2008), and in the case of analyzing frictional source terms, the definition of an extra equation assuming nil time derivative of a specific variable makes no sense.

Approximate augmented solvers, as the ARoe (Augmented Roe) solver in Murillo and García-Navarro (2010b), involve numerical strategies based on a direct discretization of all type of source terms, allowing to explain and correct if necessary, their impact in the solution. Augmented solvers allow to analyze approximate solutions involving variable density (Murillo et al., 2012), one dimensional blood flow in arteries (Murillo and García-Navarro, 2015) or applications with complex rheologies, where the well-balanced property must be redefined to provide accurate stop-and-go triggering mechanisms (Juez et al., 2014; Murillo and García-Navarro, 2011). It is remarkable that although the ARoe solver uses a limited number of characteristic waves, it still ensures convergence in resonance regions (Murillo and García-Navarro, 2012b) in the SWE.

Therefore, the ARoe solver provides a convenient way to evaluate source term discretization in situations far away from quasi-steady conditions, allowing the generation of verification