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A two-sided fractional conservation of mass equation

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ABSTRACT

A two-sided fractional conservation of mass equation is derived by using left and right fractional Mean Value Theorems. This equation extends the one-sided fractional conservation of mass equation of Wheatcraft and Meerschaert. Also, a two-sided fractional advection-dispersion equation is derived. The derivations are based on Caputo fractional derivatives.

1. Introduction

Spatial and temporal fractional-order differential equations have practical application in the modeling of hydrologic processes such as solute transport in surface water [12] and groundwater [1]. Spatially fractional equations can model rapid solute transport while temporally fractional equations can model delays in transport [8].

Various approaches have been taken for selecting the fractional governing equations for hydrologic processes. Fractional constitutive laws [2], probabilistic derivations [14] and fractional conservation laws [16] have been used. [17] discuss the various forms that the fractional advection-dispersion equation can take. Definitions and properties of fractional derivatives can be found in the works of [6] and [11].

[16] derived a fractional conservation of mass equation using the left fractional Taylor series of [10]. This conservation equation involves left local Caputo fractional derivatives and the equation is derived in a manner analogous to the derivation of the differential form of the classical conservation of mass equation. More recently, [9] used the same approach to derive a fractional Boussinesq equation.

A derivation of a fractional advection-dispersion equation using a different fractional Taylor series was presented in [13]. The series was based on a Riemann–Liouville fractional derivative. The series was used to derive a fractional Fick's Law. This fractional Fick's Law was used with a classical conservation of mass equation to derive a fractional advection-dispersion equation.

In this work, we use fractional mean value theorems to derive a two-sided fractional conservation of mass equation involving both left and right Caputo fractional derivatives. In Section 2 we discuss the fractional mean value theorems. In Section 3 we derive a two-sided conservation of mass equation. We show how the result obtained in [16] is related to the construction presented here. In Section 4 we obtain the corresponding advection-dispersion equation. We finish the paper with conclusions in Section 5.

2. Fractional mean value theorems

The left and right Caputo fractional derivatives [6] of orders $\alpha > 0$ and $\beta > 0$ of a function f can be defined by

$$({}^{L}D_a^{\alpha} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(u)}{(x-u)^{\alpha-m+1}} du \tag{1}$$

and

$$({}^{R}D_b^{\beta} f)(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_x^b \frac{f^{(m)}(u)}{(u-x)^{\beta-m+1}} du, \tag{2}$$

where $m-1 < \alpha \le m$ and $m-1 < \beta \le m$ for some positive integer m. The positions of the point of evaluation, x, and the endpoints of the interval are shown in Fig. 1.

[4] derived the following Mean Value Theorem for left Caputo fractional derivatives. For $0 < \alpha \le 1$, $f \in C[a, b]$ and ${}^{L}D_a^{\alpha} f \in C[a, b]$, there exists $\xi \in (a, b)$ such that

$$f(b) = f(a) + \frac{({}^{L}D_a^{\alpha} f)(\xi)}{\Gamma(\alpha+1)} (b-a)^{\alpha}. \tag{3}$$

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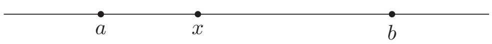

Fig. 1. The left fractional derivative of a function depends on points from a left endpoint a up to the point of evaluation x. The right fractional derivative depends on points from x up to a right endpoint b.

Fig. 2. The left fractional Mean Value Theorem shows that a function value at the right endpoint of an interval can be written in terms of the function value at the left endpoint a and ${}^{L}D_a^{\alpha} f$ at some unspecified point ξ in (a, b).

Fig. 3. The right fractional Mean Value Theorem shows that the function value at the left endpoint of an interval can be written in terms of the function value at the right endpoint b and ${}^{R}D_b^{\beta} f$ at an unspecified point θ in (a, b).

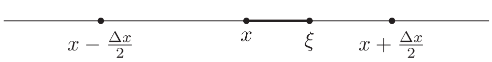

Fig. 4. The left fractional Mean Value Theorem allows $f(x+\Delta x/2)$ to be written in terms of $f(x)$ and the left fractional derivative of f at ξ.

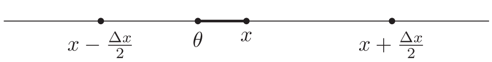

Fig. 5. The right fractional Mean Value Theorem allows $f(x-\Delta x/2)$ to be written in terms of $f(x)$ and the right fractional derivative of f at θ.

Fig. 2 shows, by using the left Mean Value Theorem, that a function value at a point b depends on the function value at a point a and the left-fractional derivative of the function at some point $\xi \in (a, b)$. The left fractional derivative of f at ξ is computed using points from a to ξ.

By analogy with the results for the left Caputo fractional derivative given in [4], similar results can be obtained for right Caputo fractional derivatives. This Mean Value Theorem for right Caputo fractional derivatives states that when $0 < \beta \le 1$, $f \in C[a, b]$ and ${}^{R}D_b^{\beta} f \in C[a, b]$, there exists $\theta \in (a, b)$ such that

$$f(a) = f(b) + \frac{({}^{R}D_b^{\beta} f)(\theta)}{\Gamma(\beta+1)}(b-a)^{\beta}. \tag{4}$$

Fig. 3 shows, by using the right Mean Value Theorem, that a function value at a point a depends on the function value at a point b and the right-fractional derivative of the function at some point $\theta \in (a, b)$. The right fractional derivative of f at θ is computed using points from θ to b.

Now consider the interval $[x-\Delta x/2, x+\Delta x/2]$. If we take $a = x$ and $b = x+\Delta x/2$ in (3) and $a = x-\Delta x/2$ and $b = x$ in (4), we obtain

$$f(x+\Delta x/2) = f(x) + \frac{({}^{L}D_x^{\alpha} f)(\xi)}{\Gamma(\alpha+1)}(\Delta x/2)^{\alpha} \tag{5}$$

and

$$f(x-\Delta x/2) = f(x) + \frac{({}^{R}D_x^{\beta} f)(\theta)}{\Gamma(\beta+1)}(\Delta x/2)^{\beta}. \tag{6}$$

When the left and right Mean Value Theorems are used, Figs. 4 and 5 show the points on which $f(x+\Delta x/2)$ and $f(x-\Delta x/2)$ depend.

Eqs. (5) and (6) are exact. We will also use approximate forms of these equations. Indeed, these fractional mean-value equations can be written in approximate form by letting ξ and θ equal the right and left endpoints of the interval, respectively

$$f(x+\Delta x/2) \approx f(x) + \frac{({}^{L}D_x^{\alpha} f)(x+\Delta x/2)}{\Gamma(\alpha+1)}(\Delta x/2)^{\alpha} \tag{7}$$

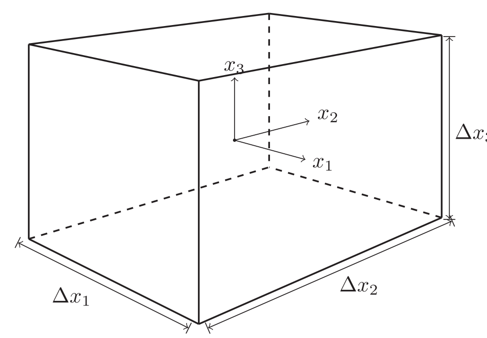

Fig. 6. Diagram of the control volume ΔV.

and

$$f(x-\Delta x/2) \approx f(x) + \frac{({}^{R}D_x^{\beta} f)(x-\Delta x/2)}{\Gamma(\beta+1)}(\Delta x/2)^{\beta}. \tag{8}$$

The fractional mean-value equations (5) and (6) can also be written in approximate form by letting ξ and θ approach the center, x, of the interval if fewer conditions on the smoothness of f are assumed. In this case, the derivatives in these equations will be local fractional derivatives:

$$f(x+\Delta x/2) \approx f(x) + \frac{({}^{L}D_x^{\alpha} f)(x+)}{\Gamma(\alpha+1)}(\Delta x/2)^{\alpha} \tag{9}$$

and

$$f(x-\Delta x/2) \approx f(x) + \frac{({}^{R}D_x^{\beta} f)(x-)}{\Gamma(\beta+1)}(\Delta x/2)^{\beta}. \tag{10}$$

The $+$ and $-$ in (9) and (10) denote the limits as the points at which the fractional derivatives are evaluated approach the center of the interval, x, from the right and left. In the Appendix, local fractional Taylor series are discussed since Eqs. (9) and (10) can also be obtained by truncating the fractional Taylor series after the second terms. Local left Caputo fractional derivatives appear in the fractional conservation of mass equation in [16] and in the derivation of the fractional Boussinesq equation in [9].

In Section 3 we use the fractional mean value theorems (5) and (6) and the approximations (7) and (8) to derive two-sided fractional conservation of mass equations.

3. Derivation of a two-sided conservation of mass equation

Now that the fractional mean value theorems have been described, we use them to derive a two-sided fractional conservation of mass equation. Consider in Fig. 6 the control volume ΔV, not necessarily infinitesimal in volume, with center at (x_1, x_2, x_3). We first describe the temporal rate of change of mass in ΔV in the x_1 direction. The resulting expression is later extended to the x_2 and x_3 directions.

Let $F_1(x_1-\Delta x_1/2, x_2, x_3, t)$ and $F_1(x_1+\Delta x_1/2, x_2, x_3, t)$ denote the components of the mass flux passing through the faces of ΔV at locations $x_1-\Delta x_1/2$ and $x_1+\Delta x_1/2$, respectively, in the x_1 direction. The temporal rate of change of mass in ΔV in the x_1 direction is then

$$F_1(x_1-\Delta x_1/2, x_2, x_3, t)A_{x_1-\Delta x_1/2} - F_1(x_1+\Delta x_1/2, x_2, x_3, t)A_{x_1+\Delta x_1/2}, \tag{11}$$

where $A_{x_1-\Delta x_1/2}$ and $A_{x_1+\Delta x_1/2}$ are the areas of the control volume faces perpendicular to the flux at locations $x_1-\Delta x_1/2$ and $x_1+\Delta x_1/2$.

We now use the exact fractional mean value theorems (5) and (6) to rewrite the flux factors F_1 at points $(x_1-\Delta x_1/2, x_2, x_3)$ and $(x_1+\Delta x_1/2, x_2, x_3)$. The values of F_1 at the faces depend on the