Advances in Water Resources 91 (2016) 122–137

Contents lists available at ScienceDirect

Advances in Water Resources

journal homepage: www.elsevier.com/locate/advwatres

One-dimensional consolidation in unsaturated soils under cyclic loading

Wei-Cheng Lo [a], Garrison Sposito [b,c], Jhe-Wei Lee [a,*], Hsiuhua Chu [d]

[a] Department of Hydraulic and Ocean Engineering, National Cheng Kung University, Tainan 701, Taiwan
[b] Department of Civil and Environmental Engineering, University of California, Berkeley, CA 94720-1710, USA
[c] Earth Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA
[d] Industrial Technology Research Institute, Information and Communications Research Laboratories, Tainan 70955, Taiwan

ARTICLE INFO

Article history:
Received 17 November 2015
Revised 28 February 2016
Accepted 1 March 2016
Available online 11 March 2016

Keywords:
Unsaturated soil
Consolidation
Cyclic loading
Poroelasticity
Dimensionless excitation frequency

ABSTRACT

The one-dimensional consolidation model of poroelasticity of Lo et al. (2014) for an unsaturated soil under constant loading is generalized to include an arbitrary time-dependent loading. A closed-form solution for the pore water and air pressures along with the total settlement is derived by employing a Fourier series representation in the spatial domain and a Laplace transformation in the time domain. This solution is illustrated for the important example of a fully-permeable soil cylinder with an undrained initial condition acted upon by a periodic stress. Our results indicate that, in terms of a dimensionless time scale, the transient solution decays to zero most slowly in a water-saturated soil, whereas for an unsaturated soil, the time for the transient solution to die out is inversely proportional to the initial water saturation. The generalization presented here shows that the diffusion time scale for pore water in an unsaturated soil is orders of magnitude greater than that in a water-saturated soil, mainly because of the much smaller hydraulic conductivity of the former.

1. Introduction

In nature, cyclic loading of porous media occurs in a variety of engineering applications, such as water level variations in seafloor sediments induced by tides [28], moving loads on pavement surfaces causing dynamic stress oscillations in the subgrade [20], and water pressure changes in a reservoir behind an earthen dam due to earthquakes [22]. Schiffman [23] seems to have been the first to derive a mathematical solution for the problem of saturated clayey soil consolidation under the action of a time-dependent load. Using the principle of superposition to combine the solutions obtained from Terzaghi's one-dimensional consolidation theory [24] for different cycling periods, Balight and Levadoux [1] presented an analytical solution of excess pore pressure and total settlement for a saturated clay layer subject to cyclic square loading. A general solution technique was proposed by Conte and Troncone [5] for the one-dimensional consolidation of a semi-permeable saturated soil under periodic loading after representing it as an infinite sum of sine and cosine wave components. Kameo et al. [10] extended Mandel's problem [26,27] to uniaxial cyclic loading, then applied the Laplace transform to derive a complete solution for the pore fluid pressure in a saturated poroelastic material [2].

In contrast to the theoretical development advanced for a saturated porous medium, scant attention has been paid to poroelastic coupling among two immiscible fluids (e.g., water and air) and the solid skeleton in an unsaturated porous medium. Fredlund and his co-workers [7,8] integrated the equations of continuity to derive constitutive equations linking stress to volume change in order to construct two coupled equations for the dissipation of pore water and air pressures under a constant external compressive load. Introducing a potential function that bears a linear relationship to the pore water and air pressures, Zhou et al. [29] solved the extended form of these equations for constant, ramped, and exponential time-dependent loading. An analytical solution was given by Ho and Fatahi [9] for the problem of two-dimensional plane consolidation employing eigenfunction expansions. In these studies or other related works undertaken to develop analytical solutions to similar problems [e.g. 6,19], the initial values for the pore water and air pressures were not state-dependent (e.g., on water saturation and porosity) parameters, but instead were constants to be determined experimentally. More recently, Lo et al. [16] applied the theory of poroelasticity to formulate exact expressions for analyzing consolidation in unsaturated soils subject to time-invariant loading. Their model provides a comprehensive theoretical

* Corresponding author. Tel.: +886 972272922.
E-mail address: jheweilee@gmail.com (J.-W. Lee).

http://dx.doi.org/10.1016/j.advwatres.2016.03.001

framework for describing the spatial and temporal evolution of solid matrix displacement, pore water pressure, and pore air pressure with well-defined initial values. Lo and Lee [17] have conducted a numerical study to examine variations in pore water pressure and total settlement as influenced by soil texture and water saturation.

Despite these advances, we do not yet have a closed-form analytical solution that describes exactly how each phase in a poroelastic medium should respond to periodic stress changes, while being rooted in a thermodynamically-consistent model which allows full coupling of the solid deformation to interstitial fluid flows. In this paper, a set of macroscopic coupled partial differential equations is developed based on the theory of poroelasticity [2,4,12,14,15] to describe one-dimensional consolidation in a partially-saturated porous medium under time-varying loading. Our equations generalize the earlier work of Lo et al. [16]. When specialized to a porous medium containing a single fluid and an elastic solid, they reduce to the well-known Biot [2] model of one-dimensional consolidation under cyclic vertical loading [26,27]. Using a Fourier series representation in the space domain [e.g. Eqs. (6)] and the Laplace transformation in the time domain [e.g. Eqs. (10)], a boundary-value problem involving our coupled model equations subject to periodic stress loading is reduced to two coupled nonhomogeneous ordinary differential equations which correspond physically to the forced vibrations of a system of coupled oscillators under viscous damping. A closed-form analytical solution that contains transient and steady-state components then is derived accounting for the spatial and temporal variations in pore water and air pressures in response to a harmonic stress excitation. This solution is illustrated through prototypical numerical simulations of pore water pressure and total settlement.

2. Model equations

We begin with an extended form of the coupled partial differential equations governing one-dimensional consolidation of an unsaturated soil, in the absence of body forces, derived by Lo et al. [16]:

$$q_1\frac{\partial p_1}{\partial t}+q_2\frac{\partial p_2}{\partial t}=b_1\frac{\partial^2 p_1}{\partial z^2}+r_1\frac{\partial f(t)}{\partial t}, \tag{1.1}$$

$$q_3\frac{\partial p_1}{\partial t}+q_4\frac{\partial p_2}{\partial t}=b_2\frac{\partial^2 p_2}{\partial z^2}+r_2\frac{\partial f(t)}{\partial t}, \tag{1.2}$$

where the coefficients q_1, q_2, q_3, q_4, b_1, b_2, and $f(t)$ are defined by

$$q_1=\left[\frac{S_1^2\alpha^2}{(K_b+\frac{4}{3}G)}-\frac{\theta_1^2 a_{33}}{(a_{23}^2-a_{22}a_{33})}\right], \tag{2.1}$$

$$q_2=q_3=\left[\frac{S_1S_2\alpha^2}{(K_b+\frac{4}{3}G)}+\frac{\theta_1\theta_2 a_{23}}{(a_{23}^2-a_{22}a_{33})}\right], \tag{2.2}$$

$$q_4=\left[\frac{S_2^2\alpha^2}{(K_b+\frac{4}{3}G)}-\frac{\theta_2^2 a_{22}}{(a_{23}^2-a_{22}a_{33})}\right], \tag{2.3}$$

$$b_1=\frac{k_s k_{r1}}{\eta_1}, \tag{2.4}$$

$$b_2=\frac{k_s k_{r2}}{\eta_2}, \tag{2.5}$$

$$r_1=\frac{S_1\alpha}{(K_b+\frac{4}{3}G)}, \tag{2.6}$$

$$r_2=\frac{S_2\alpha}{(K_b+\frac{4}{3}G)}. \tag{2.7}$$

In Eqs. (1) and (2), p_ξ and θ_ξ designate incremental gauge (or excess) pressure and volume fraction for the fluid phase ξ ($\xi=1,2$), respectively, the subscript ξ referring to a non-wetting fluid ($\xi=1$, henceforth termed "air") and a wetting fluid ($\xi=2$, henceforth termed "water"); $S_\xi=\frac{\theta_\xi}{\phi}$ denotes the relative saturation of phase ξ, ϕ being porosity; η_ξ and $k_{r\xi}$ signify its dynamic shear viscosity and relative permeability, respectively; k_s expresses the intrinsic permeability of the porous soil framework; K_b and G represent its bulk and shear moduli, respectively; $\alpha=1-\frac{K_b}{K_s}$ is the Biot–Willis coefficient [3,27], also known as the effective stress coefficient [11,27], K_s being the bulk modulus of the solid phase; and a_{22}, a_{23}, and a_{33} are linear elasticity coefficients related to directly-measurable soil properties [12,15], as summarized in Appendix A. We note in passing that in the classic consolidation problem for a fluid-containing porous medium subject to a periodic external force [26,27], the loading period is typically assumed to be much longer than the time for elastic wave propagation. Accordingly, inertial terms are negligible and the quasaistatic approximation of mechanical equilibrium in Eq. (1) remains accurate [26,27].

Eqs. (1) follow in a straightforward manner from extending Eq. (14) in Lo et al. [16] to permit a time-varying total compaction stress $f(t)$ instead of the time-invariant stress $p*$ they considered. Thus, their Eq. (14a), which represents the static equilibrium of the total stress for a three-phase system, is here extended to have the form (with the sign convention that compression is negative):

$$\left(K_b+\frac{4}{3}G\right)\frac{\partial w}{\partial z}-S_1\alpha p_1-S_2\alpha p_2=-f(t), \tag{3}$$

where w denotes a component of the displacement vector of the solid phase along the vertical (z) direction.

The physical basis of Eqs. (1) and (3) in poroelasticity theory has been discussed extensively by Lo et al. [16]. Mathematically, Eq. (1) represents two coupled nonhomogeneous diffusion equations, with the coupling occurring only in the time-derivatives. It provides a complete analytical description of one-dimensional consolidation in unsaturated soil under a cyclic load, with dependent variables p_1 and p_2. The coefficient matrix for the time-derivatives of p_1 and p_2 on the left side of Eq. (1) is symmetric. The hydraulic diffusivities [27] or consolidation coefficients [26], $\tilde{c}_v=\frac{b_1}{q_1}$ and $c_v=\frac{b_2}{q_4}$ for air and water, respectively, defined in terms of the relative mobilities b_i ($i=1,2$) in Eqs. (2.4) and (2.5) and the diagonal elements of this matrix, are crucial parameters underlying the dissipation of excess pore fluid pressure [17].

In Eq. (3), consider a sinusoidal total compaction stress with angular excitation frequency $\frac{\omega}{2}$ acting on the soil (compare Eq. (2.90) in [26]) such that a vertical surface load is applied at $t=0$ which exerts solely a compressive force, i.e., $f(t)=p^*\cos^2(\frac{\omega t}{2})$. Eq. (1) reduces to Eq. (16) (and Eq. (3) reduces to Eq. (14a)) in Lo et al. [16] if $\omega=0$. If the non-wetting fluid is absent, Eq. (1) reduces to the Biot model [2] of one-dimensional consolidation in a water-saturated porous medium under periodic vertical loading [26,27], as demonstrated in Appendix B.

3. Analytical solutions

We shall develop an analytical solution of the coupled diffusion equations in Eq. (1) for a homogeneous soil layer under the boundary conditions that the top ($z=h$) and bottom ($z=0$) surfaces are fully permeable with respect to both air and water (see Fig. 1); i.e.

$$p_1(0,t)=p_2(0,t)=0, \tag{4.1}$$

$$p_1(h,t)=p_2(h,t)=0. \tag{4.2}$$

An instantaneous undrained response (no change in air or water content) occurs throughout the soil layer when the surface stress