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A new benchmark semi-analytical solution for density-driven flow in porous media

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ABSTRACT

A new benchmark semi-analytical solution is proposed for the verification of density-driven flow codes. The problem deals with a synthetic square porous cavity subject to different salt concentrations at its vertical walls. A steady state semi-analytical solution is investigated using the Fourier–Galerkin method. Contrarily to the standard Henry problem, the cavity benchmark allows high truncation orders in the Fourier series and provides semi-analytical solutions for very small diffusion cases. The problem is also investigated numerically to validate the semi-analytical solution. The obtained results represent a set of new test case high quality data that can be effectively used for benchmarking density-driven flow codes.

1. Introduction

Numerical models are considered as irreplaceable tools for the modeling of density-driven flow in porous media [15,22,33,35,39,51,55,57]. They are used for a variety of analyses including seawater intrusion in coastal aquifers, saltwater fingering under sabkha and playa lakes, flow around salt domes as the nuclear waste repositories and saltwater upconing under freshwater lenses. The development and the use of these models require a preliminary validation step to confirm that the nonlinear governing equations are correctly solved. This step is often performed by comparing the results of the numerical models with those of existing benchmark problems. In the literature, a number of benchmarks have been proposed for density-driven flow in porous media [8,18,19,23,24,30,32,34,38,47,52]. The most popular benchmarks [56] are the Henry [23] and the Elder problems [18,19].

The Henry problem describes saltwater intrusion into a coastal aquifer. It has been widely used because of the existence of a semi-analytical solution [2–4,13,25,36,61]. This problem is considered in [56] as the sole density-driven flow problem with an exact solution. However, the semi-analytical solution of the Henry problem is limited to high values of molecular diffusion coefficient which renders it insensitive to density variations. The worthiness and benefits of this problem for benchmarking density-driven flow codes have been widely studied [16,52,53,56]. All of these studies conclude that this problem is not sufficient to test numerical models.

The Elder problem describes the flow induced by a high salinity at the top of a rectangular domain [18,19,27,42,52,53,61] and is considered to be a good benchmark for density-driven codes [56]. It is more sensitive to density variations than the Henry problem. However, due to salt instabilities and fingering phenomena, the Elder problem has no unique solution [50,52,56]. Indeed, contradictory results were reported with either upwelling or downwelling flow in the center of the domain [3,11,21,31,37]. A unique solution to the Elder Benchmark is obtained in [50] using a low Rayleigh number.

In this work, we propose a new benchmark semi-analytical solution. The problem deals with a square cavity filled with a saturated porous medium with saline water at one of its vertical walls. It is obtained by recasting the popular thermal porous cavity problem [7,9,10,44,49,54,65] as a variable-density problem where the fluid density is a function of salt concentration. A steady state semi-analytical solution is investigated for both velocity and concentration distributions using the Fourier–Galerkin (FG) method as in the Henry problem [20,23,45,48,66]. To this end, the flow and the transport equations are reformulated in terms of stream function and relative concentration. These unknowns are then expanded using appropriate Fourier series truncated at given orders. Finally, applying the Galerkin method with the Fourier terms as trial functions leads to a system of nonlinear algebraic equations having the Fourier coefficients as unknowns. This system can have a large size for small diffusion cases that

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require high truncation orders to achieve stable semi-analytical solutions [45,66]. For the Henry problem, the Fourier expansions induce fairly complex summations that hamper the development of the semi-analytical solution for these cases [14]. This difficulty is avoided in the proposed benchmark because all of the terms involving four overlapped summations are reduced to double sums. As a consequence, the semi-analytical solution is investigated for three test cases where the diffusion coefficient is taken to be, ten, one hundred and one thousand times lower than that used in the standard Henry problem. Moreover, an efficient algorithm based on the Powell hybrid method is used to solve the resulting nonlinear system of equations [40,41,43]. The efficiency of this algorithm is improved by an analytical evaluation of the Jacobian matrix.

The three test cases are also investigated numerically to assess the worthiness and benefit of the porous cavity problem for benchmarking density-driven flow codes. The numerical simulations are performed using an efficient finite element numerical model developed by Younes and Ackerer [64]. In this model, the flow equation is solved using the Mixed Hybrid Finite Element (MHFE) method [4,58,61,62] and the advection–dispersion transport equation is solved using a combination of the Discontinuous Galerkin (DG) finite element method [46,63] and the Multi-Point Flux Approximation (MPFA) method [1,59–61].

The paper is organized as follows: Section 2 is devoted to the description of the benchmark problem and its governing equations. Section 3 is devoted to the development of the semi-analytical solution. In Section 4, we show the advantages of the semi-analytical solution for the cavity problem compared to those of the Henry problem. Section 5 briefly describes the numerical solution. In Section 6, the semi-analytical and numerical results are presented and discussed in the case of high, small and very small diffusion coefficients. Finally, conclusions are provided in Section 7.

2. Benchmark description and governing equations

The benchmark is a synthetic problem inspired from the popular problem of natural convection in porous square cavity [7,9,10,44,49,54,65]. The usual thermal problem is a square box with impermeable boundaries, hot (resp. cold) right (resp. left) vertical wall and adiabatic horizontal surfaces. The solute analogous problem, considered in this work, is a square box of size H with no flow boundaries, specified concentration of value one (resp. zero) at the left (resp. right) vertical wall and zero concentration gradient on the horizontal surfaces (Fig. 1). Note that the no flow boundary condition at the left vertical wall is imposed by analogy with the thermal problem. This condition is not physically consistent for high concentrations, since contrarily to the thermal problem, the prescribed boundary concentration constitutes a source of fluid [26].

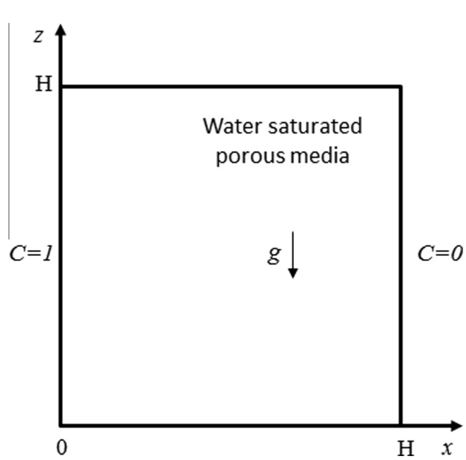

Fig. 1. Domain for the porous cavity benchmark.

The fluid flow is modeled using the continuity equation and the generalized Darcy's law. Assuming the Boussinesq approximation to be valid (i.e. the density differences are confined to the buoyancy term), these equations can be written in terms of the equivalent fresh-water head as follows [28]:

$$S\frac{\partial h}{\partial t} + \nabla\cdot\boldsymbol{q} = 0 \tag{1}$$

$$\boldsymbol{q} = -\frac{\rho_0 g}{\mu}k\left(\nabla h + \frac{\rho-\rho_0}{\rho_0}\nabla z\right) \tag{2}$$

where S is the mass storage coefficient related to the head changes $[L^{-1}]$, ρ $[ML^{-3}]$ is the density of the fluid, h is the equivalent freshwater head $[L]$, t is the time $[T]$, $\boldsymbol{q}$ is the Darcy's velocity $[LT^{-1}]$, g is the gravity acceleration $[LT^{-2}]$, μ is the dynamic viscosity of the fluid $[ML^{-1}T^{-1}]$, k is the permeability $[L^2]$, ρ_0 $[ML^{-3}]$ is the fresh water density and z is the depth $[L]$.

The solute mass transport is governed by the advection dispersion equation:

$$\frac{\partial C}{\partial t} + \boldsymbol{V}\nabla C - \nabla\cdot(\boldsymbol{D}\nabla C) = 0 \tag{3}$$

where C is the relative solute concentration $[-]$, $\boldsymbol{V} = q/\varepsilon$ is the fluid velocity, ε is the porosity and $\boldsymbol{D}$ is the dispersion tensor, reduced to molecular diffusion $\boldsymbol{D} = D_m\boldsymbol{I}$ (where $\boldsymbol{I}$ is the identity matrix).

The transport equation is coupled to the flow system via the following mixture density equation:

$$\rho = \rho_0 + (\rho_1 - \rho_0)C \tag{4}$$

where ρ_1 is the saltwater density.

The following initial and boundary conditions are used with the porous cavity problem:

$$\begin{aligned}
&t = 0: h = 0,\quad C = 0,\quad 0 \leqslant x \leqslant H,\quad 0 \leqslant z \leqslant H\\
&t > 0: q_z = 0,\quad \partial C/\partial z = 0,\quad z = 0,\quad z = H\\
&q_x = 0,\quad C = 1,\quad x = 0\\
&q_x = 0,\quad C = 0,\quad x = H
\end{aligned} \tag{5}$$

where q_x and q_z are the components of the velocity $\boldsymbol{q}$ in the x and z directions.

3. The semi-analytical solution

The steady state continuity equation implies the existence of the stream function defined by:

$$q_x = \frac{\partial\psi}{\partial z},\quad q_z = -\frac{\partial\psi}{\partial x} \tag{6}$$

The fresh water head can be eliminated using the curl of Eq. (2). To do so the horizontal and vertical components of this equation are differentiated with respect to z and x respectively, and then subtracted from each other. Using Eq. (6), the resulting equation can be written as follows:

$$\frac{\mu}{kg(\rho_1-\rho_0)}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2}\right) = \frac{\partial C}{\partial x} \tag{7}$$

Similarly, using Eq. (6), the steady state transport equation simplifies to:

$$D_m\varepsilon\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial z^2}\right) = \frac{\partial\psi}{\partial z}\frac{\partial C}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial C}{\partial z} \tag{8}$$