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Homogenization via formal multiscale asymptotics and volume averaging: How do the two techniques compare?

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ABSTRACT

A wide variety of techniques have been developed to homogenize transport equations in multiscale and multiphase systems. This has yielded a rich and diverse field, but has also resulted in the emergence of isolated scientific communities and disconnected bodies of literature. Here, our goal is to bridge the gap between formal multiscale asymptotics and the volume averaging theory. We illustrate the methodologies via a simple example application describing a parabolic transport problem and, in so doing, compare their respective advantages/disadvantages from a *practical point of view*. This paper is also intended as a pedagogical guide and may be viewed as *a tutorial for graduate students* as we provide historical context, detail subtle points with great care, and reference many fundamental works.

1. Introduction

The effective behavior of multiscale, multiphase materials has been of interest to researchers from the 19th Century. The earliest examples include Maxwell's work on the conductivity of dilute suspensions [1] and Einstein's analysis of the viscosity of a dilute suspension of neutrally buoyant hard spheres [2]. Several precursory ideas were presented in these studies, in particular the concepts of *effective* conductivity and viscosity. The continued use of these early results as limit cases or approximate correlations serves to illustrate how fundamental and remarkable they were. Nowadays, effective theories have applications as diverse as composite materials [3], biological tissues [4], biofilms [5], networks of large-scale bodies such as buildings [6], the mechanics of masonry structures [7], reservoirs with large faults [8] or transport in vascular networks [9].

A typical multiscale problem is illustrated in Fig. 1 for a porous medium. Pore-scale properties, such as the indicator field describing the phase geometry, vary *rapidly* with the spatial coordinates relative to the scales of the macroscopic domain. In Fig. 1, this is to say that the characteristic lengthscales μ and ℓ are much smaller than a characteristic large-scale length, L,

$$\mu,\ \ell \ll L. \tag{1.1}$$

Discretization of domains that satisfy Eq. (1.1) necessarily yields a substantial amount of mesh cells, making it challenging to compute solutions of partial differential equations in such multiscale systems. A solution to this numerical problem is to adopt a macroscopic viewpoint and use models in which high frequency fluctuations have been filtered out (see Fig. 2).

In most cases, such effective medium approaches were first formulated from an empirical point of view, e.g., Darcy's law [10], the dispersion equation [11–13], and the generalized Darcy's laws for multiphase flows [14]. Later on, the hypothesis that these descriptions could be obtained theoretically by averaging microscale equations found its way into the scientific community, before the sixties, and with a rapid pace thereafter. One of the first fundamental analyses related to porous media was devised in the fifties by Taylor [15] and Aris [16]. It was concerned with solute transport in a Poiseuille flow and deriving an asymptotic equation that would describe the transport of the average cross-section concentration in a tube. Taylor and Aris showed that this average satisfies a one-dimensional advection–dispersion equation and that the

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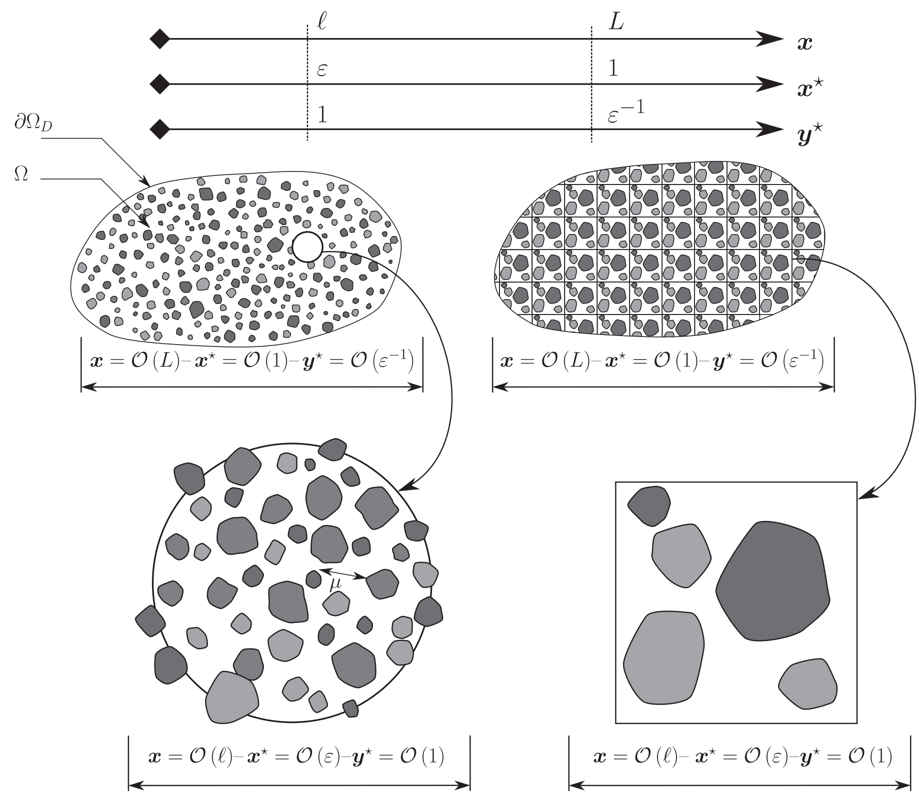

Fig. 1. Illustration of the different coordinate systems and the hierarchy of scales for *a priori* non-periodic (left-hand side) and periodic (right-hand side) media. The dimensionalized system corresponds to the spatial variable $\boldsymbol{x}$ where μ is a pore-scale characteristic length, ℓ is the size of the averaging volume and L is a macroscale characteristic length. Further, we have illustrated two additional coordinate systems that correspond to the spatial variables $\boldsymbol{x}^\star$ (macroscale) and $\boldsymbol{y}^\star$ (microscale), nondimensionalized with L and ℓ respectively.

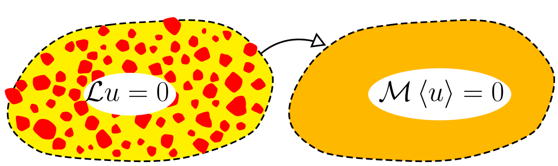

Fig. 2. Schematic diagram illustrating micro- and macroscale descriptions. The microscale differential operator, $\mathcal{L}$, applying to u, is transformed into a macroscale operator, $\mathcal{M}$, that involves effective parameters and applies to the average value $\langle u\rangle$. The microscale geometry exhibits high-frequency fluctuations that have been filtered out in the macroscale geometry.

dispersion coefficient is proportional to the square of the Péclet number. This result is valid only asymptotically (in the long-time limit), the relevant timescale being the time for a molecule of solute to travel the entire width of the tube. Hence, the analysis is particularly useful when the width of the tube is much smaller than the total length. More generally, this notion of separation of scales, Eq. (1.1), is central to the development of macroscale theories (see [17] or [18] for broad historical perspectives on mechanics).

Not only have averaging approaches led to thousands of contributions, but also a proliferation of theoretical frameworks (see [19] for a review). Generic homogenization techniques include deterministic methods such as volume averaging, multiscale asymptotics, mixture theories and the generalized method of moments (or Taylor–Aris–Brenner method, see [20]); and stochastic approaches based on ensemble averaging, i.e., where macroscale quantities are sought as mathematical expectations [21–24]. These frameworks led to significant advances in the field and to the development of new application areas such as optimal design [25] or shape optimization [26]. However, this enormous volume of works (with little connection between them) has also resulted in a lot of confusion. Indeed, how many times have we heard arguments about the relationship between the different theories, even from the most prominent contributors themselves? Surprisingly, there has been little effort to clarify these questions. One of the authors remembers hours of discussion at UC Davis with Stephen Whitaker during, or after, the visits of known contributors to porous media theories: Bourgeat, Cushman, Dagan, Gray, to cite a few. These discussions raised interesting and fundamental questions. However, on only one occasion did this lead to a public contribution: a short note (in French) comparing asymptotic homogenization and the method of volume averaging [27]. The goal that the authors outlined in this short paper remains largely unachieved and the purpose of this contribution is to advance further in this direction.

2. Historical background: volume averaging and multiscale asymptotics

2.1. Volume averaging

The idea underlying volume averaging is that macroscale variables can be defined through the use of spatial averaging. Early works in the sixties include [28–31]. For example, Marle [28,29] tried to justify Darcy's law using irreversible thermodynamics and out of equilibrium fundamental relationships (Onsager reciprocal relations). The idea that macroscale models should be compatible with thermodynamical principles was not new (see [12,32,33]), but the introduction of the volume averaging framework initiated a highly productive methodology that was