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Balancing the source terms in a SPH model for solving the shallow water equations

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ABSTRACT

A shallow flow generally features complex hydrodynamics induced by complicated domain topography and geometry. A numerical scheme with well-balanced flux and source term gradients is therefore essential before a shallow flow model can be applied to simulate real-world problems. The issue of source term balancing has been exhaustively investigated in grid-based numerical approaches, e.g. discontinuous Galerkin finite element methods and finite volume Godunov-type methods. In recent years, a relatively new computational method, smooth particle hydrodynamics (SPH), has started to gain popularity in solving the shallow water equations (SWEs). However, the well-balanced problem has not been fully investigated and resolved in the context of SPH. This work aims to discuss the well-balanced problem caused by a standard SPH discretization to the SWEs with slope source terms and derive a corrected SPH algorithm that is able to preserve the solution of lake at rest. In order to enhance the shock capturing capability of the resulting SPH model, the Monotone Upwind-centered Scheme for Conservation Laws (MUSCL) is also explored and applied to enable Riemann solver based artificial viscosity. The new SPH model is validated against several idealized benchmark tests and a real-world dam-break case and promising results are obtained.

1. Introduction

When a fluid flow problem has a vertical scale much less than its horizontal dimensions, the vertical particle acceleration may be ignored and the assumption of hydrostatic pressure becomes valid. The governing 3D Reynolds-averaged Navier–Stokes equations may then be simplified and integrated over depth to become the 2D non-linear shallow water equations (SWEs). The SWEs have been widely used in approximating the hydrodynamics of the long-wave problems and have a wide range of applications in coastal engineering, ocean modeling, river engineering and flood modeling.

Analytical solution to the SWEs only exists for very simple cases (e.g. Thacker [1]). Therefore, numerical methods are generally employed to seek approximate solutions to the SWEs. Traditionally, the SWEs are usually solved by a grid based approach including the finite difference method (FDM), the finite element method (FEM) and the finite volume method (FVM). In recent years, a robust meshless approach, smooth particle hydrodynamics (SPH), has started to gain popularity in computational fluid dynamics (CFD) and being applied in shallow flow modeling. SPH is a fully Lagrangian method proposed by Lucy [2] and Gingold and Monaghan [3]. Detailed reviews of the approach can be found in [4–6]. SPH has been applied to solve a variety of complex fluid problems including astrophysical flows [2,3], free surface flows [7–9,25] and multi-phase flows [10]. SPH has also been adopted to simulate geophysics phenomena, such as debris flows and landslides [11,12]. As a fully Lagrangian method, SPH has great potential in solving problems with free surface, deformable boundary, moving interface, which commonly occur in dam break flow, debris flow and other shallow flow phenomena. When solving these problems, grid based methods (FDM, FEM, FVM) usually face some difficulties, such as distorting and twisting of grid when handling large deformation (for Lagrangian method, e.g. FEM), difficulty to analyze the time history of field variables at a fixed point on the material (for Eulerian method, e.g. FVM) and among others [5]. Combining the advantages of the Lagrangian methods and Eulerian methods, the Arbitrary Lagrangian Eulerian (ALE) method is able to trace the time history of field variables at a fixed point on the material as well as handling large deformation. However, a numerical model

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based on the ALE method may become unstable when the deformation is significantly large [5].

In applying SPH to solve the SWEs, Wang and Shen [13] firstly applied a standard SPH formulation to solve the 1D SWEs and the numerical predictions were reported to match very well with the analytical solutions, even in presence of shocks. Ata and Soulaimani [14] proposed a stabilized SPH method to solve the SWEs with the bed slope and friction terms being neglected. Their approach was based on Riemann solvers and boasted of better shock-capturing capability. Rodriguez-Paz and Bonet [15] introduced a corrected SWE SPH scheme, in which the internal and external forces were expressed in forms of energy and the vertical velocity component was taken into account. De Leffe and Le Touzé [16] proposed an algorithm to periodically redistribute the particles in order to improve solution accuracy. However, most of these models are designed and tested for solving the SWEs that neglect the source terms and therefore are not applicable to the practical shallow flow problems where the domain topography is generally very complex.

To enable stable and accurate simulations over complex domain topographies, it is inevitable to mention the concept of 'well-balanced scheme', which essentially refers to a numerical scheme that maintains the steady-state solution, at least lake at rest, at the computational level [17]. It is important for a numerical model solving the SWEs to be 'well-balanced' in order to guarantee reliable predictions of shallow flows over complex domain topographies [17,18]. For instance, in modeling ocean waves, the water depth is usually of the order of kilometers while the waves have magnitudes of meters. The unphysical noises introduced by a numerical scheme that is not well-balanced may be of the same order as the waves and hence may completely ruin the numerical solutions. Since it was proposed in [17], the concept of 'well-balanced scheme' has been readily cited and adopted in developing grid-based shallow flow models. Intensive research efforts have been devoted in developing well-balanced shallow flow models by numerous researchers, especially in the context of finite volume Godunov-type schemes (e.g. [18–23]). However, in SPH, this problem has just started to gain attention from CFD researchers.

An initial attempt of developing a well-balanced SWE SPH model has been reported by Vacondio et al. [24]. The authors explicitly included the slope source terms into their SPH discretization. Their results indicated that the lake at rest solution in a domain with topography cannot be preserved if a special numerical treatment is not implemented. Their new model was demonstrated to reproduce the still water surface to certain level. To better resolve this problem, Vacondio et al. [37] reported more recently a corrected method for balancing the source terms. However, their model is well-balanced only under the assumption that the particles are equally distributed in space and have the same smooth length. Furthermore, their model has not been validated by a more challenging real-world problem with very complex topography.

This papers aims to investigate further the well-balanced problem in developing SPH algorithms for solving the shallow water equations with source terms, with a focus on preserving the solution of lake at rest. For this purpose, the well-balanced problem is first analyzed and discussed. The error introduced by the standard SPH approximation to the water depth over a varying bed profile is then derived and subsequently corrected to obtain a new SPH formula for both the continuity and the momentum equations, which enables the source term balancing. To enhance the shock capturing capability and increase the solution accuracy of the new corrected SPH model, the Monotone Upwind-centered Scheme for Conservation Laws (MUSCL) is applied to improve the Riemann Solver based artificial viscosity proposed by Monaghan [34]. After being demonstrated to maintain satisfactorily the still water surface of a lake at rest, the new corrected SPH model is then validated further by applying it to simulate 1D and 2D shallow flow tests over non-uniform bottom topographies and promising results are produced.

2. Brief review of the SPH method

The essence of SPH method is that any generic function can be represented as

$$f(\mathbf{x}) = \int_{\Omega} f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')d\mathbf{x}' \tag{1}$$

where Ω is the integral domain and $\delta(\mathbf{x}-\mathbf{x}')$ is the Dirac delta function defined as

$$\delta(\mathbf{x}-\mathbf{x}') = \begin{cases} \infty, & \mathbf{x}=\mathbf{x}' \\ 0, & \mathbf{x}\neq\mathbf{x}' \end{cases} \tag{2}$$

In practice, $\delta(\mathbf{x}-\mathbf{x}')$ is usually replaced by a smoothing kernel function $W(\mathbf{x}-\mathbf{x}',h)$ with h being the smoothing length that determines the influence domain of the kernel. Hence Eq. (1) becomes

$$f(\mathbf{x}) \approx \int_{\Omega} f(\mathbf{x}')W(\mathbf{x}-\mathbf{x}',h)d\mathbf{x}' \tag{3}$$

which is called the kernel approximation. In the context of SPH, kernel approximation operator is usually marked with <> and so Eq. (3) can be rewritten as

$$\langle f(\mathbf{x})\rangle = \int_{\Omega} f(\mathbf{x}')W(\mathbf{x}-\mathbf{x}',h)d\mathbf{x}' \tag{4}$$

The kernel approximation of the gradient of $f(\mathbf{x})$ can be obtained by

$$\langle \nabla f(\mathbf{x})\rangle = \int_{\Omega} f(\mathbf{x}')\nabla W(\mathbf{x}-\mathbf{x}',h)d\mathbf{x}' \tag{5}$$

There are several mathematical conditions that the kernel function must satisfy, which are

(i) $\lim_{h\to 0} W(\mathbf{x}-\mathbf{x}',h) = \delta(\mathbf{x}-\mathbf{x}',h)$
(ii) $\int_{\Omega} W(\mathbf{x}-\mathbf{x}',h)d\mathbf{x}' = 1$, which ensures the zero-order consistency.
(iii) $W(\mathbf{x}-\mathbf{x}',h) = 0, |x-x'| > \kappa h$ where κ is an integer (usually <4)
(iv) $W(\mathbf{x}-\mathbf{x}',h) = W(\mathbf{x}'-\mathbf{x},h)$

In SPH calculations, it is practical to subdivide the integral domain Ω into a finite set of N particles. Each particle carries a mass of m_j and has a density of ρ_j, with $j = 1,2,3,\ldots,N$. The sum of the mass for every particle gives the total mass of the fluid body under consideration. The SPH approximation of $f(\mathbf{x})$ and $\nabla f(\mathbf{x})$ at $\mathbf{x}_i$ can then be calculated by replacing the integral with the summation of respective quantities of the contributing particles, i.e.

$$f_i = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f_j W_{ij} \tag{6}$$

$$\nabla f_i = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f_j \nabla W_{ij} \tag{7}$$

where $W_{ij} = W(\mathbf{x}_i - \mathbf{x}_j, h)$. Since $W(\mathbf{x}-\mathbf{x}',h) = 0$ when $|x-x'| > \kappa h$, the total number of the contributing particles is actually much less than N.

In this work, a Gaussian kernel [5] is adopted, which leads to a second-order accurate approximation. Detailed discussion of other kernel functions may be found in [5].