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Ekman circulation and downwelling in narrow lakes

Marco Toffolon *

Department of Civil and Environmental Engineering, University of Trento, via Mesiano 77, 38123 Trento, Italy
Eawag, Swiss Federal Institute of Aquatic Science and Technology, Kastanienbaum, Switzerland

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ABSTRACT

An analytical solution is obtained for the wind-driven steady flow developing under the action of the Coriolis acceleration in a closed basin of elongated shape. Different from the traditional Ekman approach, which determines the velocity distribution along a water column given the free surface shear stress and pressure gradient, here the flow field is solved in the whole cross-section considering the lateral transfer of momentum due to the horizontal eddy viscosity. The solution is derived exploiting a perturbation method, whereby the inverse of the Ekman number is assumed small, and imposing a wind aligned with the main axis of the lake. In the central part of the lake a secondary circulation develops producing downwelling along the right hand side (in the northern hemisphere) and upwelling along the opposite side, whose intensity is modulated by the turbulence anisotropy. The modification of the primary flow is considered as well. The solution, which is also compared with numerical results, is obtained for simplified conditions, but the extension to more general cases is discussed.

1. Introduction

The traditional approach in describing the effect of Coriolis acceleration on wind-driven circulation in lakes and oceans dates to Ekman's pioneering contribution [1] and consists of the determination of the vertical profile of the horizontal components of the velocity vector. The so-called Ekman spiral, whereby the flow velocity changes direction along the vertical in the region where the Coriolis force is balanced by the vertical transfer of momentum, was thought as a steady solution for the open ocean. In fact, far from lateral boundaries, the vertical variation of the horizontal velocity can be determined solely as a function of the local values of the wind shear stress at the surface and the pressure gradient [2]. While the former parameter is the actual source of momentum, which is transferred downward by turbulent diffusion, the latter is a result of the flow field. Nevertheless, the free surface slope is often assumed as an input for the vertical structure, and is determined by means of suitable two-dimensional depth-averaged models [2–4], usually structured in the form of a vorticity equation. Such a depth-averaged approach intrinsically requires a specific treatment of the boundary layer regions close to the shores, where a vertical current develops to satisfy the three-dimensional continuity equation. This current is characterised by streamlines that become more vertical as the slopes get steeper, and are not accounted for in the standard Ekman solution.

* Address: Department of Civil and Environmental Engineering, University of Trento, via Mesiano 77, 38123 Trento, Italy.
E-mail address: marco.toffolon@ing.unitn.it

In this paper we adopt a different approach: we directly solve the flow field in an entire cross-section orthogonal to the wind direction without separating the vertical dependence from the horizontal depth-averaged variables. In this way, there is no explicit need to introduce boundary layers at the lateral sides. The new approach is specifically relevant for the case of narrow basins with steep slopes. In this sense, the proposed solution is complementary to the recent contribution by Winant [3]: there, the case of a mildly variable bathymetry was tackled by splitting the vertical problem from the horizontal depth-integrated dynamics, and neglecting horizontal viscosity; here, we obtain a fully explicit analytical solution by solving the coupled horizontal-vertical problem in a cross-section exploiting the assumption that the lateral sides are vertical. This latter type of basin morphology is a reasonable approximation of fjords and lakes with steep slopes (as many lakes in alpine regions are), which therefore constitute the main object of the present analysis.

In the new solution, the effect of Earth rotation is not vertically limited by the frictional dissipation, as for the Ekman layer in the open ocean. On the contrary, it generates a secondary circulation (resembling that occurring in estuaries), which affects the whole depth. The actual region where the streamlines are closed depends on the stratification of the water body: for simplicity, we will consider the whole water column of a homogeneous basin. However, analogous considerations about the developing secondary circulation can be extended to the superficial mixed layer (epilimnion) of a thermally stratified lake.

It is important to note that a large number of studies are already available about the effect of Coriolis acceleration on the various

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internal waves produced by the interaction with the boundaries in natural lakes and other closed basins (e.g. Kelvin waves), and on the motion in the Ekman superficial layer in the oceans (e.g., [5]). Here we investigate only the steady response to a constant and homogeneous wind forcing, which is of course not a novel problem (e.g., [6,3]). However, differently from the analyses developed so far, the horizontal eddy viscosity is accounted for, as in [7], and its role is shown to be significant both for the mathematical solution and for the assessment of the correct intensity of the circulation. In fact, retaining the horizontal viscosity allows one to find a proper analytical solution for the flow field in a closed basin without the need to impose an arbitrary structure of the forcing wind to satisfy the boundary conditions, as previous solutions required (see [4] for a review).

The paper is structured as follows: the simplified governing equations are introduced in Section 2, the method to derive the analytical solution is presented in Section 3, and the linearised flow field is illustrated in Section 4. The comparison with the numerical solution of the complete nonlinear problem, the extension of the results to more general cases and an example concerning a real lake are considered in Section 5.

2. Formulation of the problem

Looking for an analytical solution obviously requires several assumptions. In the following analysis we consider a simplified rectangular domain having longitudinal length L_x^* (an asterisk denotes dimensional quantities), lateral width L_y^*, and constant depth D_0^* (Fig. 1). In particular, we assume that $L_x^* \gg L_y^*$, i.e. that the shape of the basin is elongated, as it is often found in real alpine lakes, and that the wind is directed along the main axis x^* with uniform and constant velocity. Moreover, the variation of the Coriolis parameter f^* with latitude is neglected, a reasonable assumption for the relatively small domains typical of lake and fjord studies.

Since the purpose of this paper is to highlight the main features of a physical process by retaining only the main mechanisms, we aim at simplifying the governing equations as much as possible. As a starting point, we consider the standard formulation for geophysical flows (e.g., [5,8]), which is composed of the three-dimensional Reynolds averaged equations in the shallow water approximation. In fact, it is common to assume (also for deep lakes) that the vertical distribution of pressure is approximately hydrostatic, since the vertical velocity is usually small enough to avoid significant effects of pressure modifications on the overall circulation (differently from the case of some types of internal

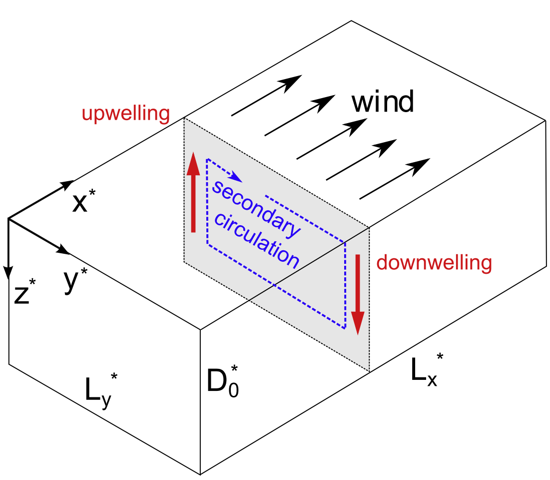

Fig. 1. Sketch of the simplified basin with idealised clockwise secondary circulation.

waves). Furthermore, as a first approximation of the real problem, we consider the density as uniform, i.e. we assume that the compartment we are interested in, consisting of either the whole lake or the epilimnion, is well mixed. Hence the two horizontal momentum equations and the continuity equation are written, without considering baroclinic effects, in the form

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*} + f^* v^* + g^*\frac{\partial \eta^*}{\partial x^*} = \frac{\partial}{\partial x^*}\left(\nu_x^*\frac{\partial u^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(\nu_y^*\frac{\partial u^*}{\partial y^*}\right) + \frac{\partial}{\partial z^*}\left(\nu_z^*\frac{\partial u^*}{\partial z^*}\right), \tag{1}$$

$$\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*} - f^* u^* + g^*\frac{\partial \eta^*}{\partial y^*} = \frac{\partial}{\partial x^*}\left(\nu_x^*\frac{\partial v^*}{\partial x^*}\right) + \frac{\partial}{\partial y^*}\left(\nu_y^*\frac{\partial v^*}{\partial y^*}\right) + \frac{\partial}{\partial z^*}\left(\nu_z^*\frac{\partial v^*}{\partial z^*}\right), \tag{2}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0, \tag{3}$$

where u^*, v^* and w^* are the components of the velocity vector along the longitudinal (x^*), lateral (y^*) and vertical (z^*) axes, t^* is time, η^* is the free surface elevation, g^* is gravity acceleration, and ν_x^*, ν_y^*, ν_z^* are the eddy viscosity coefficients.

There is a long-standing debate about the tensorially correct geophysical treatment of the frictional terms [4], but the simple Fickian closure of the Reynolds turbulent stresses is sufficient for our aims [8]. Focusing on the flow field in a single layer (i.e., a well-mixed compartment), we further assume that ν_x^*, ν_y^* and ν_z^* are homogeneous and that, typically, $\nu_x^* \sim \nu_y^* \gg \nu_z^*$ [7]. The extension to more layers is possible but requires additional algebra (see [7] for an example). The assumption of constant eddy diffusivity coefficients, or of an arbitrarily assigned vertical profile, represents a first approximation needed to reasonably limit the amount of algebra [4,7], disclosing however the main dynamics. Clearly, the vertical variation of these coefficients can be crucial for the hydrodynamics, and will surely need future attention.

In the adopted notation, the vertical coordinate z^* is directed downward with origin at the free surface. This choice allows for plotting the results conveniently in the y^*–z^* cross section keeping a coordinate system with a right-handed orientation. Note that, because of the downward direction of z^*, the usual sign of terms proportional to f^* in (1)–(2) are inverted (hereafter, we will always refer to the northern hemisphere, where $f^* > 0$ gives a deviation to the right of the flow).

Eqs. (1)–(3) can be written in dimensionless form introducing suitable scales:

$$x = \frac{x^*}{L_x^*}, \quad y = \frac{y^*}{L_y^*}, \quad z = \frac{z^*}{D_0^*}, \quad \{u, v\} = \frac{\{u^*, v^*\}}{U_0^*}, \quad w = \frac{D_0^*}{U_0^* L_y^*} w^*, \tag{4}$$

where U_0^* is the scale of velocity, and defining the free surface slopes as follows

$$\frac{\partial \eta^*}{\partial x^*} = s_x, \quad \frac{\partial \eta^*}{\partial y^*} = s_y. \tag{5}$$

Provided that $L_x^* \gg L_y^*$, we consider only the central part of the lake, where all longitudinal gradients can be neglected apart from the slope of the free surface [7]. In this 'trunk' region [9], a secondary circulation can develop in the y–z plane. Note that the scale for w^* in (4) suggests that the vertical velocity is much smaller than the horizontal components, typically by a ratio D_0^*/L_y^* that comes from the consideration that the vertical gradient of velocity in (3) can be balanced only by the lateral gradient when the longitudinal gradient vanishes. Hence, when a steady state has been achieved, the governing equations for momentum and continuity become (see also Appendix A)