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Auxiliary Variable-based Balancing (AVB) for source term treatment in open channel simulations

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ABSTRACT

Practical engineering applications of open channel flow modelling involve geometric terms arising from variations in channel shape, bottom slope and friction. This paper presents the family of schemes that satisfy the generalised *C*-property for which static equilibrium is a particular case, in the framework of one-dimensional open channel flows. This approach, named Auxiliary Variable-based Balancing, consists of using an auxiliary variable in place of the flow variables in the diffusive part of the flux estimate. The auxiliary variable is defined so as to achieve a zero gradient under steady-state conditions, whatever the geometry. Many approaches presented in the literature can be viewed as a particular AVB case. Three auxiliary variables are presented in this paper: water elevation, specific force and hydraulic head. The methodology is applied to three classical Riemann solvers: HLL, Roe and the *Q*-scheme. The results are compared on five test-cases: three steady-state configurations including friction, singular head losses and variations in bottom elevation, channel width and banks slope and two transient test-case (dam-break problems on rectangular and triangular channel). In each case, the auxiliary variable that best preserves the steady-state configuration is the hydraulic head. Besides, using the head as auxiliary variable allows head loss functions due to singularities to be incorporated directly in the governing equations, without the need for internal boundaries. However, it is generally less accurate when sharp transients are involved.

1. Introduction

In hydrodynamic modelling, real-world applications of computational open channel simulations involve the discretization of source terms arising from bottom slope, non-prismatic channel, etc. Attempting to discretize the fluxes and source terms independently from each other usually leads to stability problems. An indispensable prerequisite is that the discretization of flux gradients and geometric source terms should allow static equilibrium conditions to be preserved. This is known as the *C*-property [4,38]. The need for source term discretization techniques that preserve equilibrium conditions without introducing spurious oscillations in the computed variables has led to the general notion of well-balanced schemes. Over the past two decades, substantial research effort has been devoted to the influence of source terms discretization techniques [32] and new definitions that preserve the *C*-property, including applications to high-order schemes such as WENO (weighted essentially non-oscillatory) methods (e.g. [7,10,13,39]).

The various existing source term discretization approaches may be classified into two broad families: (i) approaches where the source term discretization technique is adapted to the flux formulae, and (ii) approaches where the flux formulae are adapted from, or derived in a coupled way with, the source term discretization. Examples of the former approach are source term upwinding [4,38] and derived techniques such as predictor–corrector [3] or introduction of the source terms in the flux formulation [9], divergence form for the bed slope source term (DBF) [37], the quasi-steady wave propagation method [29], asymptotic balancing [12] or the source term projection technique in discontinuous Galerkin techniques [27]. Examples of the latter are the well-balanced approach [1,2,8,24,31,35], flux and source term splitting [11], characteristics-based approximate-state and augmented Riemann solvers [10,16,18,20,30], the homogeneous approach [28] and other static equilibrium-preserving techniques [6,19,40].

Various solutions have also been proposed to enforce the *C*-property in finite volume-based discretizations. One of the earliest solutions, proposed in [33] for the solution of the SWE and later extended in [40], consists in replacing the water depth with the free surface elevation. This option can be extended to the open channel equations in arbitrary-shaped channels, as shown in the present paper. It has the drawback that simple flow configurations such as uniform flow over a constant slope cannot be computed accurately (see Section 3.2). Another option is to approximate the variations in the cross-sectional area with a consistent estimate

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taken from the balance between the specific force and the source term in the momentum equation [6]. The estimate is defined in such a way that it is zero under steady state conditions. Very similar formulae to that of [6] have been obtained using completely different approaches in [28,30]. The approaches [6,28,30] have the common point that the gradient in one of the flow variables is replaced with the gradient in another variable, called auxiliary variable hereafter. This gradient is zero under static conditions. That different approaches yield the same formulae lead to wonder whether a general methodology can be derived to define auxiliary variables.

In the present paper, the principle of Auxiliary Variable-based Balancing (AVB) is presented for one-dimensional free surface flow calculations in non-prismatic, trapezoidal channels. This is motivated by the fact that in industrial open channel packages, the cross-sectional geometry is broken into a set of trapezoidal elements. The AVB approach is used to derive flux formulae that allow non-static, steady state flow conditions to be preserved, even at low orders of discretization, that is, when first-order schemes are used.

The principle of the AVB method is presented for the water hammer and one-dimensional SWE in [25]. However, the one-dimensional shallow water equations are a very simplified description of free surface flows in natural channels. Besides, only one possible approach for source term discretization (a variant of source term upwinding) is considered in [25]. The applicability of the approach to more complex cross-sections and other source term discretization approaches is not investigated in [25]. The objectives of the present paper are (i) to present the methodology of Auxiliary Variable-based Balancing (AVB), (ii) to apply the AVB approach to the open channel flow equations in a well-balanced, finite volume framework, (iii) to provide the flux and source term discretizations for a variety of Riemann solvers, and (iv) to analyse the accuracy of the numerical solutions obtained using a number of various AVB-based discretizations. As mentioned above, first-order space discretizations are retained for the sake of computational rapidity.

The structure of the paper is as follows. Section 2 presents the governing equations and their discretisation. The AVB methodology is detailed in Section 3 and its application to classical Riemann solvers presented in Section 4. Section 5 provides computational examples, including steady-state configurations and transient test-cases as well as a convergence analysis for the classical dam-break problem (for which an analytical solution is available).

2. Governing equations and solution method

2.1. Governing equations

The purpose is to solve 2×2 hyperbolic systems of conservation laws in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S} \tag{1}$$

where **U**, **F** and **S** are defined as

$$\mathbf{U} = \begin{bmatrix} A \\ Q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} Q \\ M \end{bmatrix} = \begin{bmatrix} Q \\ \frac{Q^2}{A} + \frac{P}{\rho} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ (S_0 - S_f)gA + \frac{R}{\rho} \end{bmatrix} \tag{2}$$

where A is the cross-sectional area, g is the gravitational acceleration, M is the specific force, P is the pressure force exerted on the wetted cross-sectional area, S_0 and S_f are respectively the bottom and energy slope, R is the x-component of the reaction of the walls onto the water (if the channel is non-prismatic) and ρ is the water density.

The forces P and R are derived from the assumption of a hydrostatic pressure distribution and obey the following definitions [14]:

$$\frac{P}{\rho} = \int_A (\zeta - z) g \, \mathrm{d}A = \int_0^h (h - z') g W(z') \, \mathrm{d}z' \tag{3}$$

$$\frac{R}{\rho} = \int_0^h (h - z') g \left(\frac{\partial W}{\partial x} \right)_{h - z' = \mathrm{Const}} (z') \, \mathrm{d}z' \tag{4}$$

where $W(z)$ is the width of the channel at the elevation z, h is the water depth (that is the distance between the lowest point in the cross-section and the free surface), $z' = z - z_b$ is the elevation above the bottom lowest point and ζ is the free surface elevation (Fig. 1).

The energy slope is classically assumed to obey a turbulent-type friction law such as Manning's law:

$$S_f = n_M^2 u^2 R_H^{-4/3} \tag{5}$$

where n_M is Manning's friction coefficient, $u = Q/A$ is the flow velocity and R_H is the hydraulic radius, defined as the ratio of the cross-sectional area A to the wetted perimeter χ, yielding

$$S_f = n_M^2 Q^2 A^{-10/3} \chi^{4/3} \tag{6}$$

It is noted that the Jacobian matrix **A** of **F** with respect to **U** is given by

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix} \tag{7}$$

where the speed c of the waves in still water is defined as

$$c^2 \equiv \frac{\partial \left(\frac{P}{\rho} \right)}{\partial A} = \frac{gA}{b} \tag{8}$$

where $b = W(\zeta)$ is the top width of the channel. The matrix **A** can be diagonalized into a matrix $\mathbf{\Lambda}$ defined as:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{bmatrix} \tag{9a}$$

$$\lambda^{(1)} = u - c \tag{9b}$$

$$\lambda^{(2)} = u + c \tag{9c}$$

The problem is assumed to be properly posed hereafter, that is, the initial and boundary conditions are specified such that Eq. (1) can be solved uniquely for **U** at all points of a computational domain $[0, L]$ for all times $t > 0$.

2.2. Finite volume discretization

Eq. (1) is discretized using a finite volume formalism as

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x_i} \left(\mathbf{F}_{i-\frac{1}{2}}^{n+\frac{1}{2}} - \mathbf{F}_{i+\frac{1}{2}}^{n+\frac{1}{2}} \right) + \Delta t \mathbf{S}_i^{n+\frac{1}{2}} \tag{10}$$

where the subscript i denotes a cell average, subscripts $i \pm \frac{1}{2}$ denote estimates at the interfaces between the computational cells, the

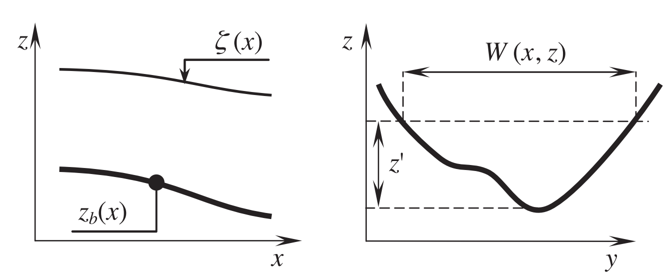

Fig. 1. Channel geometry. (Left) longitudinal view: bottom and water elevation. (Right) transversal view: channel width and depth.