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Exact analytical solutions for two-dimensional advection–dispersion equation in cylindrical coordinates subject to third-type inlet boundary condition

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ABSTRACT

Exact analytical solutions for two-dimensional advection–dispersion equation (ADE) in cylindrical coordinates subject to the third-type inlet boundary condition are presented in this study. The finite Hankel transform technique in combination with the Laplace transform method is adopted to solve the two-dimensional ADE in cylindrical coordinates. Solutions are derived for both continuous input and instantaneous slug input. The developed analytical solutions are compared with the solutions for first-type inlet boundary condition to illustrate the influence of the inlet condition on the two-dimensional solute transport in a porous medium system with a radial geometry. Results show significant discrepancies between the breakthrough curves obtained from analytical solutions for the first-type and third-type inlet boundary conditions for large longitudinal dispersion coefficients. The developed solutions conserve the solute mass and are efficient tools for simultaneous determination of the longitudinal and transverse dispersion coefficients from a laboratory-scale radial column experiment or an in situ infiltration test with a tracer.

1. Introduction

The advection–dispersion equation (ADE) has been widely used to describe the movement of contaminants in the subsurface. Analytical solutions for ADE play important roles in giving initial or approximate estimates of contaminant concentration distributions in soil or aquifer systems, for determining transport parameters, and for validating more comprehensive numerical models. A number of analytical solutions for one-, two- and three-dimensional ADEs have been developed for predicting the transport of various contaminants in the subsurface. For example, van Genuchten and Alves [1] formulated several analytical solutions for the one-dimensional ADE subject to various initial and boundary conditions. Batu [2,3] presented analytical solutions to two-dimensional ADE with various source boundary conditions. Leij et al. [4] and Park and Zhan [5] derived analytical solutions for three-dimensional ADE. Although a number of two- or three-dimensional analytical models are available at present, these models are mostly limited to the case of the ADE in Cartesian coordinates with steady uniform flow [5].

Analytical solutions for two-dimensional ADE in cylindrical coordinates are particularly useful for analyzing problems of the two-dimensional solute transport in a porous medium system with a radial geometry associated with a circular source. For example, they can be applied to interpret the field infiltration test with a tracer [6] or a two-dimensional laboratory column transport experiment [7,8] for simultaneous determining the longitudinal and transverse dispersion coefficients. Leij et al. [4] derived an analytical solution for two-dimensional ADE in cylindrical coordinates subject to the first- and third-type conditions using Laplace and Hankel transform technique by considering the medium is infinite in flow direction and in radial direction. Massabó et al. [9] has derived some analytical solutions to the two-dimensional ADE in cylindrical coordinates subject to the first-type condition using successive applications of the Laplace transform and the Fourier–Bessel series expansion by considering the medium is infinite in flow direction and finite in radial direction. Similar to Leij et al. [4], Zhang et al. [6] developed an analytical solution to the two-dimensional ADE in cylindrical coordinates but considered a third-type instantaneous slug input. It should be noted that the analytical solutions derived by Massabó et al. [9] are based on the first-type inlet boundary condition which specifies a given solute concentration at the surface of the column. Previously work has shown that analytical solution for the first-type and third-type boundary conditions can be associated with flux- and volume-average concentration, respectively [10–13]. Analytical solution for the first-type inlet condition gives rise to physically improper mass conservation and significant errors in predicting the solute concentration distribution especially for a porous medium system with a large longitudinal dispersion coefficient if it is interpreted to

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represent the usual volume-average concentration. Selection of the appropriate inlet boundary condition has been the subject of much research such as solute transport either in a uniform flow or a radial flow field [14–20].

The purpose of this study is to develop exact analytical solutions for two-dimensional ADE in cylindrical coordinates subject to the third-type inlet boundary condition. The finite Hankel transform technique coupled with the Laplace transform method is applied to obtain the exact analytical solutions to the two-dimensional ADE in cylindrical coordinates. The finite Hankel transform technique provides a systematic, efficient and straightforward approach for obtaining analytical solutions for both transient and steady flow transport problems with a radial geometry. The developed solutions are compared with the analytical solutions for the first-type inlet boundary condition to illustrate the impact of the inlet boundary condition.

2. Mathematical model

This study considers the problem of two-dimensional solute transport in a porous media system with a radial geometry and a circular source as illustrated schematically in Fig. 1. The flow is steady and uniform along the downward column axis. The solute mass is injected over the inner zone of the upper surface. The injected solute migrates in the z direction by advection and longitudinal dispersion, whereas it spreads in the r direction by transverse dispersion. Solute transport in such a radial system can be described by ADE in cylindrical coordinates as

$$D_L \frac{\partial^2 C}{\partial z^2} - V\frac{\partial C}{\partial z} + \frac{D_T}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right) = \frac{\partial C}{\partial t} \tag{1}$$

where $C(z,r,t)$ denotes the solute concentration; t is the time; V stands for the averaged steady-state pore water velocity; and D_L and D_T represent the longitudinal and transverse dispersion coefficients, respectively.

The system is free of solute at the initial time:

$$C(z,r,t=0) = 0 \quad 0 \leqslant z \leqslant \infty,\ 0 \leqslant r \leqslant R \tag{2}$$

where R is the radius of the radial system.

The solute mass is applied over the inner zone of the upper surface. Based on mass conservation across the upper surface of the column system, the continuous input can be formulated as

$$VC(r,z=0,t) - D_L\frac{\partial C(r,z=0,t)}{\partial z} = \begin{cases} VC_0 & 0 \leqslant r \leqslant \rho \\ 0 & \rho \leqslant r \leqslant R \end{cases} \tag{3a}$$

and the instantaneous slug input as

$$VC(r,z=0,t) - D_L\frac{\partial C(r,z=0,t)}{\partial z} = \begin{cases} \frac{\sigma}{\phi}\delta(t) & 0 \leqslant r \leqslant \rho \\ 0 & \rho \leqslant r \leqslant R \end{cases} \tag{3b}$$

where C_0 is the concentration of the applied solute solution (assumed to be constant) for continuous input; ρ is the radius of the inner inlet zone; $\sigma = \frac{M}{\pi\rho^2}$ is the solute mass distribution over the inner inlet zone for the instantaneous input; M is the injected solute mass.

Another boundary condition for variable z required for obtaining a unique solution to Eq. (1) is imposed at infinity as follows:

$$C(r,z\to\infty,t) = 0 \tag{4}$$

The boundary condition (4) indicates that the radial system is assumed to be of semi-infinite length.

The boundary condition at $r = R$ assumes impermeable condition as

$$\frac{\partial C(z,r=R,t)}{\partial r} = 0 \tag{5}$$

3. Derivation of the analytical solution

The aforementioned two-dimensional ADE in cylindrical coordinates and its associated initial and boundary conditions are analytically solved by successive implementation of the finite Hankel transform and the Laplace transform.

First, utilizing the second kind of finite Hankel transform with Eq. (1) with respect to r results in

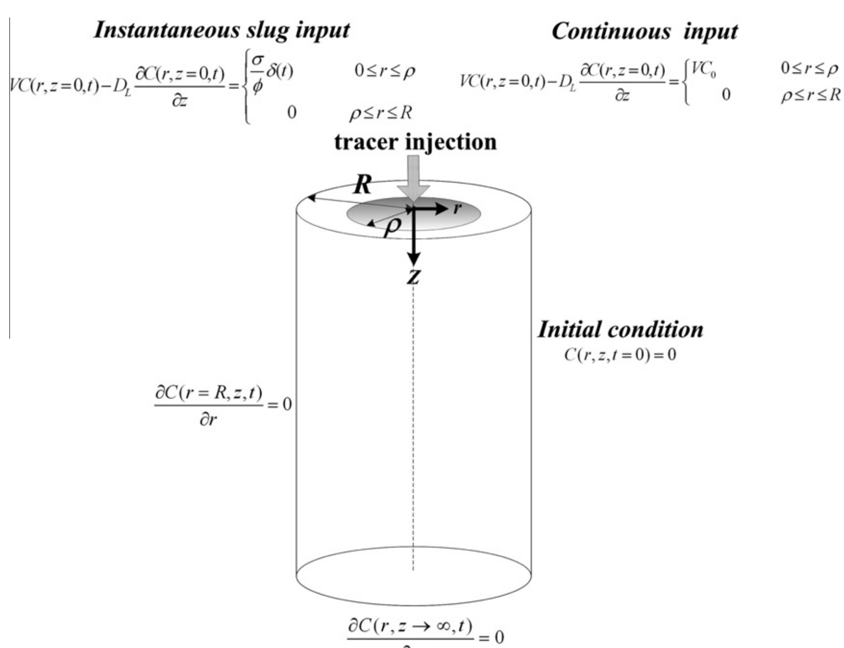

Fig. 1. Schematic representation of the two-dimensional advective–dispersive transport in a porous medium system with cylindrical geometry subject to steady and uniform flow field. The injected solute mass is applied over the inner zone of the upper surface.