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# Definition and interests of reciprocity and reciprocity gap principles for groundwater flow problems

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ABSTRACT

We introduce the reciprocity and reciprocity gap principles for flow problems in hydrogeology and illustrate their interest in addressing identification problems. The reciprocity principle is derived from mechanics and establishes for flow problems a relationship between different sets of forcing terms, including sources, sinks and boundary conditions, and the resulting head fields. The reciprocity gap principle compares different head fields resulting from the same forcing terms applied to different structures. We give general 2D expressions of the reciprocity and reciprocity gap principles for transient flow problems and give two examples of applications for the identification of transmissivity values and interfaces between different transmissivities. Identification capacities of the reciprocity and reciprocity gap principles yielding direct inversion methods could be used as initial guesses for more advanced inverse problem methodologies.


## 1. Introduction

The reciprocity gap principle is built up on the classical reciprocity principle also known as the Maxwell-Betti theorem [7,8], first introduced in mechanics for linear problems [2,3,5,10]. According to the reciprocity principle for a linear elastic structure subjected to two forces F and G, the work resulting from the application of force F on the displacement field yielded by force G is equal to the work resulting from the application force G on the displacement field yielded by force F. The reciprocity principle is classically applied in mechanics to obtain displacements due to complex forces by using simpler proxy problems with simpler forces that are more easily solved [14]. From a phenomenological perspective, it establishes strong relationships between different sets of forces and the consequent displacements applied to a given structure. When the structure is altered, a gap is introduced in the reciprocity. Reciprocity breaks down but the gap within the reciprocity becomes highly informative about the alteration of the structure and can be used inversely to identify the solid alteration. The bottom line of identification then consists in comparing through the reciprocity gap the response of the altered domain with the response of a simpler reference [2,3]. The reciprocity gap yields directly an optimization problem. Very generally, the interests of both the reciprocity and reciprocity gap principles strongly depend on the ability of defining the simpler proxy problems and simpler references mentioned above, which will be generically referred to in the remaining of the text as the test functions.

Similar reciprocity relationships have been derived for slow viscous flows from Lorentz reciprocal theorem [17] and from the energy balance to establish Darcy's law from pore-scale mass transport and energy equations [19,21]. A practical analogy can be established with the reciprocity principle of mechanics by using sources and boundary conditions as forcing terms and the resulting head field as a consequence. It is no longer the work of the forces that is conserved but the viscous dissipation energy. We argue in this paper that the simple reciprocity and the reciprocity gap principles and their concealed identification possibilities are of interest for groundwater flows. After a general review of the reciprocity and reciprocity gap principles (Section 2), we derive them for groundwater flows (Section 3) and illustrate their interests in parameter and structure identifications (Section 4).

## 2. Basics on the reciprocity and reciprocity gap principles

### 2.1. General formulation and interpretation of the reciprocity principle

We express the reciprocity principle in a general mathematical framework for linear problems. Let $\mathcal{V}$ be a Hilbert space associated to a

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domain $\Omega$, $\mathfrak{a}$ a bilinear form on $\mathcal{V}$, assumed to be symmetric, continuous and coercive and $\mathfrak{l}_i$ a linear form defined on $\mathcal{V}$ for $i=1, 2$ assumed to be continuous. We define the following variational problem :

$$\text{Find } u_i \text{ on } \mathcal{V} \text{ such that } \quad \mathfrak{a}(u_i, \varphi) = \mathfrak{l}_i(\varphi) \text{ for all } \varphi \in \Omega. \tag{1}$$

With $(u_i, \varphi)$ successively equal to $(u_1, u_2)$ and $(u_2, u_1)$ and using the symmetry of operator $\mathfrak{a}$, the reciprocity principle can be expressed by the identity:

$$\mathfrak{l}_1(u_2) = \mathfrak{l}_2(u_1). \tag{2}$$

The reciprocity principle is fundamentally derived from the symmetry and the bilinearity of the form $\mathfrak{a}$. From a physical perspective, it relates the responses to different external and internal forcing terms (source/sink terms, boundary conditions) of a given phenomenon on a fixed structure represented by form $\mathfrak{a}$. The reciprocity principle is also known as the virtual work principle where $\varphi$ are called virtual fields, and as the variational or weak formulation where $\varphi$ are called test functions. Finally, the reciprocity principle is similar to Green's second identity giving way to the boundary element methods sometime used as an alternative numerical method to the Finite Element and Finite Difference methods. For diffusion problems, the boundary element methods use the Green functions as test functions with inner sources (i.e. source inside or at the limit of the computational domain) [9,11].

### 2.2. From reciprocity to reciprocity gap

When operator $\mathfrak{a}$ is modified to $\mathfrak{a}_T$ by altering the geometry or the parameters of the structure, the reciprocity principle does no longer hold. Where the reciprocity principle characterizes the consequences of forcing terms, the reciprocity gap is a signature of alteration in the domain structure or in the parameters. We take $\mathfrak{a}$ as the reference problem and express $\mathfrak{a}_T$ in the same kind of formulation as Eq. (1) for $\mathfrak{a}$:

$$\mathfrak{a}_T(u_T, v) = \mathfrak{l}_T(v)$$

where $u_T$ and $v$ are two functions of the Hilbert space $\mathcal{V}$. The reciprocity gap is expressed by:

$$\mathfrak{a}_T(u, u_T) - \mathfrak{a}(u_T, u) = \mathfrak{l}_T(u) - \mathfrak{l}(u_T) \tag{3}$$

The reciprocity gap principle has been initially introduced in the mechanical context for flaws detection. It has also been used for boundary data completion where the objective is to recover missing boundary data when the boundary conditions are missing in a part of the boundary and over-determined in the remaining part of the boundary [6]. It has also been applied to the inverse scattering problem in order to determine the shape of an unknown object embedded in an otherwise homogeneous domain from data at the domain limits [12].

### 2.3. Consistent choice of test functions with regard to targeted applications

The critical point of the method is the relevant choice of the test functions. The test functions should ideally be closely related to the initial problem but should lead to much simpler and, if possible, analytical solutions. For example, for Laplace equation, classical test functions are harmonic polynomials and Green's functions. Harmonic polynomials in 2D are real or imaginary part of $z^k$ with $k$ a positive or negative integer and $z$ the complex variable taken with an origin outside of the computational domain [13,18]. Green's functions are $(-log(\|x - x_0\|)$ in 2D with $x_0$ outside of the computational domain [1]. Using again the terminology of the alteration of a known system, test functions give some moments of the alteration. In a more sophisticated approach, test functions could be built incrementally for non-homogeneous but simpler reference media. The reciprocity gap would then compare known simple to unknown more complex media [4].

## 3. Reciprocity and reciprocity gap principles applied to the 2D flow equation

We derive the reciprocity gap principle for general transient Darcean single-phase flows in porous media. Combining continuity equations and Darcy's law integrated on the vertical dimension, we derive the following boundary value problem that governs the hydraulic head $h$ in a 2D aquifer over a time interval $[t_0, t_f]$:

$$\begin{cases} S(x,y)\frac{\partial h}{\partial t} - \nabla.(T(x,y)\nabla h) = Q & \text{in } \Omega \times [t_0, t_f], \quad (a) \\ T\nabla h.n = Q_N & \text{on } \Gamma_N \times [t_0, t_f], \quad (b) \\ h = h_D & \text{on } \Gamma_D \times [t_0, t_f], \quad (c) \\ h(x,y,t_0) = h_0 & \text{in } \Omega \quad (d) \end{cases} \tag{4}$$

with $S(x, y)$ the storage coefficient field, $T(x, y)$ the transmissivity field, $Q$ the internal source/sink terms, $h_D$ the prescribed heads at the Dirichlet boundary $\Gamma_D$, $Q_N$ the prescribed flux at the remaining Neuman boundary $\Gamma_N$ and $h_0$ the initial hydraulic head distribution. We hereafter establish successively the reciprocity gap expression for transient flows for generic 2D media and for the specific 2D zonation case.

### 3.1. Reciprocity principle for general 2D flow problems

Reciprocity can be applied to the transient flow problem using test functions derived from the adjoint of the flow Eq. (4-a) $\varphi$ defined by :

$$\begin{cases} S(x,y)\frac{\partial \varphi}{\partial t} + \nabla.(T(x,y)\nabla \varphi) = 0 & \text{in } \Omega \times [t_0, t_f], \\ T\nabla\varphi.n = \Psi & \text{on } \partial\Omega \times [t_0, t_f], \\ \varphi(x,y,t_f) = \varphi_f & \text{in } \Omega \end{cases} \tag{5}$$

where $\varphi$ is the test function, $\Psi$ is a flow boundary condition at the full limit $\partial\Omega = \Gamma_N \cup \Gamma_N$ and $\varphi_f$ is the final head distribution. We note that we can set whatever type of boundary conditions for the test function $\varphi$. We have used here the Neuman boundary conditions for getting not too complex expressions of the reciprocity principles.

Multiplying the first equations of systems (4) and (5) by $\varphi$ and $h$ respectively, integrating them over $\Omega$, and applying Green's first identity lead to the two following equations:

$$\begin{aligned} &\int_\Omega S(x,y)\frac{\partial h}{\partial t}.\varphi + \int_\Omega T(x,y)\nabla h.\nabla\varphi - \int_{\partial\Omega} T(x,y)\frac{\partial h}{\partial n}.\varphi = \int_\Omega Q\varphi \\ &\int_\Omega S(x,y)\frac{\partial \varphi}{\partial t}.h - \int_\Omega T(x,y)\nabla h.\nabla\varphi + \int_{\partial\Omega} T(x,y)\frac{\partial \varphi}{\partial n}.h = 0. \end{aligned} \tag{6}$$

Summing these two equations and integrating them over the time range $[t_0, t_f]$ leads to the transient-state expression of the reciprocity principle for the flow equation:

$$\int_\Omega S\Big(h(t_f).\varphi_f - h_0.\varphi(t_0)\Big) = \int_{t_0}^{t_f}\int_{\partial\Omega}\Big(T\frac{\partial h}{\partial n}.\varphi - h.\Psi\Big) + \int_{t_0}^{t_f}\int_\Omega Q\varphi. \tag{7}$$

It relates the head values at the boundary at any time, the head values in the domain at the initial and final times with the transmissivity and storage coefficient fields.