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Quasilinear infiltration from an elliptical cavity

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ABSTRACT

We develop analytic solutions to the linearized steady-state Richards equation for head and total flowrate due to an elliptic cylinder cavity with a specified pressure head boundary condition. They are generalizations of the circular cylinder cavity solutions of Philip [Philip JR. Steady infiltration from circular cylindrical cavities. Soil Sci Soc Am J 1984;48:270–8]. The circular and strip sources are limiting cases of the elliptical cylinder solution, derived for both horizontally- and vertically-aligned ellipses. We give approximate rational polynomial expressions for total flowrate from an elliptical cylinder over a range of sizes and shapes. The exact elliptical solution is in terms of Mathieu functions, which themselves are generalizations of and computed from trigonometric and Bessel functions. The required Mathieu functions are computed from a matrix eigenvector problem, a modern approach that is straightforward to implement using available linear algebra libraries. Although less efficient and potentially less accurate than the iterative continued fraction approach, the matrix approach is simpler to understand and implement and is valid over a wider parameter range.

1. Introduction

A solution for flow from a long elliptic cylinder cavity is given in two-dimensional elliptical coordinates for the quasilinear [26] form of the steady unsaturated flow equation [31] in a homogeneous porous medium. The solution is an extension of one by Philip [27] for flow from a circular cylinder cavity.

The approach taken here is to expand the linearized potential in the natural eigenfunctions that arise in elliptical coordinates. This technique has been utilized extensively in the physics literature (e.g., [2,7,15,22,24,34], but the solution derived here for the current problem's boundary conditions is new.

Unsaturated porous media flow, specifically infiltration, is a very non-linear process that is often solved numerically with finite element codes such as HYDRUS-2D (e.g., [32]). Analytic solutions to infiltration problems, restricted as they may be, often deliver more insightful results due to their simplicity. They give solutions with fewer potentially complicating auxiliary parameters. Pullan [29] reviews the history of the quasilinear solution methodology and compares numerous approaches for solving the linearized Richards equation.

In the context of predicting furrow infiltration, Rawls et al. [30] compared steady infiltration solutions for 1, 2, and 3 dimensions, using the 2D point source solution of Philip [26] in the comparison. The solution derived here for an elliptical shape is more realistically furrow-shaped; ellipses have the capability of simulating the geometry associated with either wide or deep cavities and strips. Warrick et al. [38] and Warrick and Lazarovitch [37] discuss the impacts that dimensionality and "edge effects" have on infiltration from strips and parabolic-shaped furrows.

The elliptical solution derived here can represent the geometry of a strip or furrow explicitly, although without surface or water table boundary effects. It is a free-space solution, since it is valid at large distance. A dry far-field condition is assumed, resulting in no-flow far away from the ellipse. Including the effects of the land surface (potentially intersecting the ellipse) would require imposing a no-flow boundary condition. This homogeneous type II boundary condition would become an inhomogeneous type III boundary condition after applying the required non-linear transformations [40]. A solution for flow from an elliptical cavity that accounted for this boundary condition would most likely be approximate in nature (e.g., a linearized AEM or gridded numerical solution). An alternative approach would be to use the integral expression of Lomen and Warrick [18, Eq. (5)] (with $D = 0$, and no dependence on Y or T) to include the effects of a horizontal evaporative or no-flow boundary. Similarly, Philip [28] and Warrick [36, p. 276] indicate how to account for a water table condition beginning with a known free-space solution. Using the solution derived here in these integral relationships leads to integral expressions that cannot be evaluated in closed form for general values of the coordinates.

Bakker and Nieber [4] applied the analytic element method to the quasilinear flow equation for the problem of uniform vertical

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flow through ellipses of different material properties. Their approach is quite general, but to obtain a solution for multiple elements involves performing two nested iterations. A non-linear boundary-matching iteration is nested within an outer iteration that accounts for the effects elements have on one another. In the analysis presented here, no iterations are required to compute the solution, outside of those potentially needed to compute the required Mathieu functions (also needed for the AEM solution).

Mathieu functions are the special functions that arise as solutions to the Helmholtz equation in elliptic-cylinder coordinates [3,5,23,24]. We use a modern matrix eigenvector approach [6,33], allowing all the required functions and coefficients to be computed using any combination of widely available eigensolution (e.g., Matlab [20] or LAPACK [12]) and Bessel function routines.

2. Mathematical formulation

2.1. Quasilinear governing equation

The steady-state unsaturated porous media flow equation [31] is

$$\widehat{\nabla} \cdot (K(h)\widehat{\nabla} h) = \frac{\partial K}{\partial z}, \tag{1}$$

where $K(h)$ is hydraulic conductivity $[L/T]$, a non-linear function of pressure head, $h[L]$. Flow is driven by gradients in hydraulic head, $\Phi = h - z$, the sum of pressure and elevation heads (z positive down). Hats indicate dimensional differential operators. The Kirchhoff transformation [16] is used to linearize (1); it is

$$\Theta(h) = \int_{-\infty}^{h} K(u)\,\mathrm{d}u, \tag{2}$$

where u is a dummy variable and Θ is matric flux potential $[L^2/T]$. Applying (2) leads to

$$\widehat{\nabla}^2 \Theta = \frac{1}{K}\frac{\mathrm{d}K}{\mathrm{d}h}\frac{\partial \Theta}{\partial z}. \tag{3}$$

The Gardner [10] exponential $K(h)$ distribution is used to simplify (3) further, by assuming the linearizing relationship $K(h) = K_0 \mathrm{e}^{\alpha h}$, where $h < 0$ for unsaturated flow, α is the sorptive number $[1/L]$ (related to pore size) and K_0 is K at saturation. With the Gardner distribution, (3) becomes

$$\widehat{\nabla}^2 \Theta = \alpha \frac{\partial \Theta}{\partial z}, \tag{4}$$

the steady quasilinear form of Richards' equation. The quasilinear approximation was first extensively studied by Philip [26]. Pullan [29] summarizes its benefits and limitations.

2.2. Elliptical geometry

A long elliptical pipe is represented as a surface of constant radius in two-dimensional elliptic-cylinder coordinates, where the variation along the length of the pipe is negligible. For a horizontal ellipse, the major axis is parallel to the land surface and the positive z-axis points down (see Fig. 1). The elliptical angular coordinate starts at the positive x-axis and increases clockwise, $0 \leqslant \psi \leqslant 2\pi$. Cartesian coordinates $(x, z)[L]$ are defined in terms of the dimensionless elliptical coordinates (η, ψ) as

$$x = f\cosh(\eta)\cos(\psi), \quad z = f\sinh(\eta)\sin(\psi), \tag{5}$$

where f is the semi-focal distance $[L]$; the cylindrical boundary is $\eta = \eta_0$. The eccentricity of the ellipse is a dimensionless quantity

$$e = \sqrt{1 - \frac{b^2}{a^2}}, \tag{6}$$

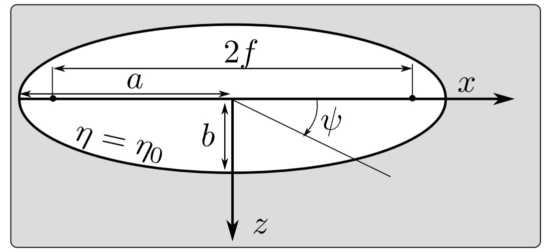

Fig. 1. Elliptical coordinates η (radial) and ψ (angular); a, b, and f are the semi-major, -minor, and -focal lengths.

equivalently $f = ea$, that ranges from 0 (circle) to 1 (line segment joining the foci). The pair (a, e) completely specifies the geometry of the problem; a is a measure of the size of the cavity, while e is related to its shape. The circumference of the ellipse, $c[L]$, cannot be evaluated exactly in closed form. It is defined by an elliptic integral, but can be approximated using one of several formulas. We use the simple YNOT expression [19]

$$c \approx 4\sqrt[y]{a^y + b^y}, \tag{7}$$

where $y = \ln(2)/\ln(\frac{\pi}{2})$ and the error in the approximation is at most 0.4%.

2.3. Non-dimensionalizing

Because of the problem's homogeneity, it can be made dimensionless with respect to the porous medium's sorptive number. Dimensionless lengths are defined

$$\frac{A}{a} = \frac{B}{b} = \frac{F}{f} = \frac{C}{c} = \frac{X}{x} = \frac{Z}{z} = \frac{\alpha}{2}, \tag{8}$$

where capital letters are dimensionless versions of lower-case variables. The dimensionless matric flux potential is $\vartheta = \Theta/\Theta(\eta_0)$. The boundary condition on the ellipse is specified pressure head, $h(\eta_0) = h_0$. For simplicity, the far-field boundary condition is no-flow

$$h(\eta \to \infty) = -\infty, \quad \Theta[h(\eta \to \infty)] = 0, \tag{9}$$

this corresponds to an assumption of dry conditions away from the cutout, i.e., the flow field is dominated by the moisture infiltrating from the ellipse.

The linearized flow equation (4) in dimensionless form, after the exponential substitution $\vartheta = H\mathrm{e}^{Z}$, becomes the Yukawa [9] or modified Helmholtz equation

$$\nabla^2 H = H, \tag{10}$$

subject to the boundary conditions

$$H(\eta_0, \psi) = \mathrm{e}^{-F\sinh(\eta_0)\sin(\psi)} = \mathrm{e}^{-B\sin(\psi)}, \tag{11}$$

$$H(\eta \to \infty) = H\mathrm{e}^{F\sinh(\eta)\sin(\psi)} \to 0 \tag{12}$$

in elliptical coordinates. Many solutions to (10) are available in the physics literature, though the combination of boundary condition (11) and elliptical geometry makes the solution derived here unique.

The dimensionless pressure ($\boldsymbol{\Psi}$) and hydraulic ($\boldsymbol{\Phi}$) heads are defined and related to ϑ by

$$\boldsymbol{\Psi} = \frac{h}{h - h_0} = \frac{1}{2}\ln(\vartheta), \quad \boldsymbol{\Phi} = \frac{\Phi}{\Phi - h_0} = \frac{1}{2}\ln(\vartheta) - Z, \tag{13}$$

where we take $h_0 = 0$ for simplicity.