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Three-dimensional semi-analytical solution to groundwater flow in confined and unconfined wedge-shaped aquifers

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ABSTRACT

The Laplace domain solutions have been obtained for three-dimensional groundwater flow to a well in confined and unconfined wedge-shaped aquifers. The solutions take into account partial penetration effects, instantaneous drainage or delayed yield, vertical anisotropy and the water table boundary condition. As a basis, the Laplace domain solutions for drawdown created by a point source in uniform, anisotropic confined and unconfined wedge-shaped aquifers are first derived. Then, by the principle of superposition the point source solutions are extended to the cases of partially and fully penetrating wells. Unlike the previous solution for the confined aquifer that contains improper integrals arising from the Hankel transform [Yeh HD, Chang YC. New analytical solutions for groundwater flow in wedge-shaped aquifers with various topographic boundary conditions. Adv Water Resour 2006;26:471–80], numerical evaluation of our solution is relatively easy using well known numerical Laplace inversion methods. The effects of wedge angle, pumping well location and observation point location on drawdown and the effects of partial penetration, screen location and delay index on the wedge boundary hydraulic gradient in unconfined aquifers have also been investigated. The results are presented in the form of dimensionless drawdown-time and boundary gradient-time type curves. The curves are useful for parameter identification, calculation of stream depletion rates and the assessment of water budgets in river basins.

1. Introduction

Wedge-shaped aquifers form in a variety of geological conditions. For example, a groundwater flow system with a wedge-shaped aquifer is commonly formed by an ancient alluvial fan [20]. In river basins with several tributaries, wells in alluvial valleys are often located near the wedge-shaped confluence of two tributaries [9]. In general, the solution for the drawdown distribution in a wedge-shaped aquifer with various boundaries may be obtained by introducing imaginary (or image) wells. However, the image theory is applicable only when the angle between the bounding radii can be expressed as $360/n$, where n is an integer and 360 is a multiple of n [15].

Kuo et al. [8] utilized the image-well method to predict the drawdown distribution in aquifers with irregularly shaped boundaries; however, their solutions may diverge if insufficient number and improper locations of the image wells are employed. Chan et al. [3] developed an analytical line source solution for drawdown in rectangular aquifers. Instead of using the image-well method, they applied a finite double Fourier transform to obtain their solution. Carslaw [1] developed an analytical point source and line source solutions for heat conduction, which is analogous to groundwater flow, in a wedge-shaped domain. Chan et al. [2] developed an analytical solution for a confined wedge-shaped aquifer with zero drawdown boundaries. They applied a finite sine transform and a Hankel transform to obtain steady state and transient solutions for an infinite wedge. They also applied a finite sine transform and a Mellin transform to obtain a steady state solution which does not contain an infinite series term and is much easier to evaluate. Instead of using zero drawdown boundaries, Yeh and Chang [20] derived an analytical wedge-shaped aquifer solution with various topographic boundary conditions. Like Chan et al. [2], they applied a Hankel transform to develop their solution and showed that the Chan et al. [2] solution is a special case of their solution. Yeh and Chang [20] further simplified their solution using available mathematical tables and using Shank's [16] method to facilitate the numerical evaluation of their solution. Yeh et al. [21] used Yeh and Chang [20] solution to calculate stream depletion rate and volume for two semi-finite streams that intersect at a point with an arbitrary angle. All of the above solutions are obtained for two-dimensional confined aquifers and to our knowledge there are no analytical solutions applicable to unconfined wedge-shaped aquifers.

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Nomenclature

d	aquifer thickness (m)
h	hydraulic head (m)
I_a	modified Bessel function of first kind and order a
K_a	modified Bessel function of second kind and order a
K_h	horizontal hydraulic conductivity (m/s)
K_z	vertical hydraulic conductivity (m/s)
Q	discharge (m^3/s)
r_0, θ_0, z_0	coordinate of sink/source point
r, θ, z	coordinate of observation point
r_{0D}, θ_0, z_{0D}	dimensionless coordinate of sink/source point
r_D, θ, z_D	dimensionless coordinate of observation point
$r_{D<}$	smaller one of r_{0D} and r_D
$r_{D>}$	larger one of r_{0D} and r_D
s	drawdown (m)
s_D	dimensionless point source drawdown in a piezometer
s_{pD}	dimensionless drawdown due to a partially penetrating well in a piezometer
s_{opD}	dimensionless drawdown due to a partially penetrating well in a screened observation well
S_s	specific storativity (m^{-1})
S_y	specific yield
t	time (s)
t_D	dimensionless time defined in Table 1
t_{Dy}	dimensionless time defined in Table 1
$W(f1, f2, x_0)$	Wronskian between $f1$ and $f2$ at x_0
α_1	empirical constant for drainage from unsaturated zone
$\delta(u)$	Dirac delta function
φ	wedge angle (radian)
$\Gamma(x)$	Gamma function

The objective of this paper is to present Laplace domain solutions for drawdown in three-dimensional, homogeneous, anisotropic, confined and unconfined, wedge-shaped aquifers. At first, a fundamental solution for drawdown due to a point source in a uniform anisotropic aquifer is derived. Then the point source solution is extended to linear sinks that represent the partially or fully penetrating wells. In the case of an unconfined aquifer the effect of delay yield is taken into account. It has been mathematically shown that the late time approximation of our solution for the confined aquifer is the steady state solution of Chan et al. [2]. Our solution also is verified with the image method for wedge angles that satisfy the $360/n$ criterion. Previous wedge-shaped aquifer solutions contain integrals that are numerically difficult to evaluate. The Laplace domain solution presented here is easily evaluated using well known numerical Laplace inversion algorithm of Stehfest [18].

2. Groundwater flow to a well in a wedge-shaped aquifer

Fig. 1 is schematic diagrams of the confined and unconfined wedge-shaped aquifers coordinate system setup with a partially penetrating well. The coordinate system (r, θ, z) is a three-dimensional cylindrical system. The r-axis and θ-axis are horizontal and z-axis is vertical. The origin is at the intersection point of two boundaries at $z = 0$ and $r = 0$ and z direction is positive upward. The lateral boundaries are zero drawdown (constant head). Hydraulic conductivity is assumed to be the same in all horizontal directions. In the following, the solution for transient flow to a point source is derived and then extended to flow to a linear source that represents a well.

The governing equation for transient three-dimensional groundwater flow to a point source in an anisotropic wedge-shaped aquifer is:

$$K_h\left(\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}+\frac{1}{r^2}\frac{\partial^2 h}{\partial \theta^2}\right)+K_z\frac{\partial^2 h}{\partial z^2}-S_s\frac{\partial h}{\partial t} = \frac{Q}{r}\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0) \quad (1)$$

subject to the following initial and boundary conditions:

$$h(r,\theta,z,t=0)=h_0 \quad \text{(The initial condition)} \quad (2)$$

$$h(r,\theta=0,z,t)=h(r,\theta=\varphi,z,t)=h_0 \quad \text{(Lateral constant head boundaries)} \quad (3)$$

$$h(r\to\infty,\theta,z,t)=h_0 \quad \text{(Lateral constant head boundaries at far distance)} \quad (4)$$

$$\frac{\partial h}{\partial z}(r,\theta,z=0,t)=0 \quad \text{(No-flow condition at the base of the aquifer)} \quad (5)$$

$$\frac{\partial h}{\partial z}(r,\theta,z=d,t)=0 \quad \text{(No-flow condition at the top of the aquifer)} \quad (6)$$

For the unconfined aquifer Eq. (6) will be replaced by the following boundary condition related to the delayed yield, approximated by Moench [10]:

$$K_z\frac{\partial h}{\partial z}(r,\theta,z=d,t)=-\alpha_1 S_y\int_0^t \frac{\partial h(r,\theta,z=d,t')}{\partial t'}\exp(-\alpha_1(t-t'))dt' \quad (7)$$

where S_s is the specific storativity (m^{-1}); h, the hydraulic head (m); t, the time (s); K_h, K_z (m/s), the hydraulic conductivity in horizontal and vertical directions, respectively; Q the pumping rate (m^3/s) ($Q > 0$ for pumping and $Q < 0$ for injection); δ, is the Dirac delta function (m^{-1}); h_0, the initial hydraulic head (m); d, aquifer thickness (m); φ, the wedge angle (radian); S_y is the specific yield; α_1 is the experimental delay index and (r_0, θ_0, z_0) is the source location. The point source included as a Dirac delta function δ in Eq. (1). It should be noted that Eq. (1) is an extension of the governing equation for two-dimensional transient groundwater flow to a line source in a confined wedge-shaped aquifer provided by Chan et al. [2].

For convenience, the hydraulic head h can be changed to drawdown $s = h_0 - h$. Using the dimensionless parameters in Table 1, the dimensionless form of Eqs. (1)–(7) in the Laplace domain becomes:

$$\frac{\partial^2 \bar{s}_D}{\partial r_D^2}+\frac{1}{r_D}\frac{\partial \bar{s}_D}{\partial r_D}+\frac{1}{r_D^2}\frac{\partial^2 \bar{s}_D}{\partial \theta^2}+\frac{\partial^2 \bar{s}_D}{\partial z_D^2}-p\bar{s}_D = -\frac{4\pi}{pr_D}\delta(r_D-r_{0D})\delta(\theta-\theta_0)\delta(z_D-z_{0D}) \quad (8)$$

$$\bar{s}_D(r_D,\theta=0,z_D,p)=\bar{s}_D(r_D,\theta=\varphi,z_D,p)=0 \quad (9)$$

$$\bar{s}_D(r_D\to\infty,\theta,z_D,p)=0 \quad (10)$$

$$\frac{\partial \bar{s}_D}{\partial z_D}(r_D,\theta,z_D=0,p)=0 \quad (11)$$

$$\frac{\partial \bar{s}_D}{\partial z_D}(r_D,\theta,z_D=1,p)=0 \quad (12)$$