Ocean Modelling 101 (2016) 113–120

Contents lists available at ScienceDirect

Ocean Modelling

journal homepage: www.elsevier.com/locate/ocemod

Short communication

Conventional versus pre-balanced forms of the shallow-water equations solved using finite-volume method

Xinhua Lu[a,*], Shengbai Xie[b]

[a] State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, China
[b] College of Ocean, Earth and Environment, University of Delaware, Newark, Delaware 19716, USA

ARTICLE INFO

Article history:
Received 6 July 2015
Revised 26 March 2016
Accepted 8 April 2016
Available online 9 April 2016

Keywords:
FORCE
HLLC
Pre-balanced
Shallow-water equations
SLIC
Well-balanced

ABSTRACT

In the existing literature, various forms of governing equations have been proposed to solve the shallow-water equations (SWEs). Recently, attention has been dedicated to the so-called "pre-balanced" form, because finite-volume schemes that are designed on this basis satisfy the well-balanced property. In this study, we theoretically investigate the relationship between numerical schemes devised using approximate Riemann solvers in the framework of finite-volume methods for solving the conventional form of the SWEs and its "pre-balanced" variant. We find that the numerical schemes for solving these two forms of the SWEs turn out to be identical when some widely employed upwind or centered approximate Riemann solvers are adopted for the numerical flux evaluations, such as the HLL (Harten, Lax, and van Leer), HLLC (HLL solver with restoring the contact surface), FORCE (first-order centered), and SLIC (slope limited centered) schemes. Some numerical experiments are performed, which verify the validity of the result of our theoretical analysis. The theoretical and numerical results suggest that the "pre-balanced" SWEs variant is not superior to the conventional one for solving the SWEs using approximate Riemann solvers.

1. Introduction

In regions where the water depth is far less than the horizontal scale of motion, the shallow-water equations (SWEs) are widely employed for the modelling of flow motions (e.g., flood wave propagation, wave run-ups, wind-induced flow motions), scalar transport (by coupling the SWEs with a scalar transport equation), and sediment transport and the associated morphological processes (by coupling the SWEs with sediment transport and bed deformation equations). In Fig. 1, a sketch of the shallow-water system is presented. The SWEs, in a conservative vector form, can be written as (see, for example, Liang and Borthwick (2009))

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial \boldsymbol{G}}{\partial y} = \boldsymbol{S}, \tag{1}$$

with vectors defined by

$$\boldsymbol{U} = \begin{bmatrix} z \\ q_x \\ q_y \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} F_c \\ F_x \\ F_y \end{bmatrix} = \begin{bmatrix} q_x \\ F_x \\ uq_y \end{bmatrix},$$

$$\boldsymbol{G} = \begin{bmatrix} G_c \\ G_x \\ G_y \end{bmatrix} = \begin{bmatrix} q_y \\ vq_x \\ G_y \end{bmatrix}, \quad \boldsymbol{S} = \boldsymbol{S}_f + \boldsymbol{S}_0 = \begin{bmatrix} 0 \\ S_{fx} \\ S_{fy} \end{bmatrix} + \begin{bmatrix} 0 \\ S_{0x} \\ S_{0y} \end{bmatrix}, \tag{2}$$

where t denotes time; x and y denote the horizontal Cartesian coordinates; z denotes the water surface elevation above a horizontal reference level z_r; h denotes the water depth; and $q_x = uh$ and $q_y = vh$ denote discharges per unit width, with u and v denoting the depth-averaged velocity components in the x- and y-directions, respectively. The terms with subscripts "f" and "0" denote the bed friction and water surface (or bed elevation) gradient related forces, respectively. Throughout this paper, variables in a non-bold style with subscripts "c", "x", and "y", respectively, are related to the continuity equation and the momentum equations in the x- and y-directions, and these variables are the first, second, and third components of a vector. The bed friction force may be calculated using Manning's formula, as

$$S_{fx} = -\frac{gn^2 q_x \sqrt{q_x^2 + q_y^2}}{h^{7/3}}, \quad S_{fy} = -\frac{gn^2 q_y \sqrt{q_x^2 + q_y^2}}{h^{7/3}}, \tag{3}$$

where n denotes Manning's roughness coefficient. Note that except z, h could also be used as the conserved variable for the continuity equation (first components of Eqs. (1) and (2)).

Various forms of the SWEs exist in the literature. They mainly differ in the expressions of F_x and S_{0x} in the x-direction and G_y

* Corresponding author at: State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, China.
E-mail address: xhluhh@whu.edu.cn (X. Lu).

http://dx.doi.org/10.1016/j.ocemod.2016.04.002
1463-5003/

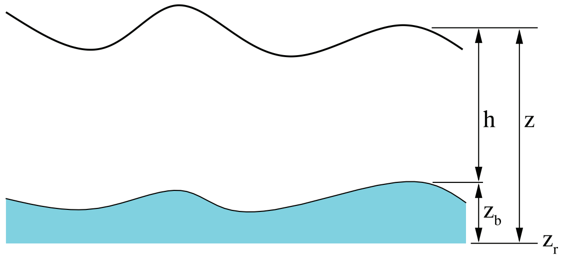

Fig. 1. Sketch of a shallow-water system (not to scale).

and S_{0y} in the y-direction in Eq. (2). For instance, F_x and S_{0x} are commonly written as (Toro, 2001)

$$F_x = uq_x + \frac{1}{2}gh^2, \tag{4a}$$

$$S_{0x} = -gh\frac{\partial z_b}{\partial x}, \tag{4b}$$

where z_b denotes the bed elevation above z_r and physically, z_b satisfies the equality

$$z = h + z_b. \tag{5}$$

For the convenience of descriptions to follow, Eqs. (1) and (2) with F_x and S_{0x} as defined by Eq. (4) will be referred to as the HH form of the SWEs, or simply SWE-HH, in reference to the fact that the expression of F_x (and G_y) contains the term h^2. Other forms of the SWEs can be found in the literature, among which the so-called "pre-balanced" form proposed by Liang and Borthwick (2009) has drawn great attention. Here, a numerical scheme is called well-balanced (e.g., satisfy the C-property) if the discretizations of the flux gradient and source terms in the momentum equations are consistent, such that under a quiescent flow condition over a complex bed topography, no spurious flow is predicted (Bermudez and Vazquez, 1994). Liang and Borthwick (2009) reformulated SWE-HH such that the flux gradient and source terms are perfectly balanced when the HLL (Harten, Lax, and van Leer) approximate Riemann solver (Harten et al., 1983) is used for the numerical flux evaluations. Note that this pre-balanced form of the SWEs was derived specifically for Godunov-type finite-volume schemes, where the relevant expressions of F_x and S_{0x} are defined as

$$F_x = uq_x + \frac{g}{2}z(z - 2z_b), \tag{6a}$$

$$S_{0x} = -gz\frac{\partial z_b}{\partial x}; \tag{6b}$$

the corresponding SWEs will be referred to as the ZZ form of the SWEs, or SWE-ZZ, in viewing of the fact that the expression of F_x (also G_y) contains the term z^2.

Since it was first proposed, the ZZ form of the SWEs has become popular, and has been adopted by many researchers to solve the SWEs over a complex topography (see, for example, Kesserwani and Liang (2010); Huang et al. (2013); Vacondio et al. (2014); Qian et al. (2015)). The idea of pre-balancing by reformulating the HH form of a governing equation system to a ZZ form has been widely adopted to process the flux gradient and source terms in some other equation systems, in which the flux gradient and source terms are the same as those in the SWEs. Such equation systems include the Boussinesq-type equations (Ning et al., 2008), the Green–Naghdi equations (Duran and Marche, 2015; Lannes and Marche, 2015), the SWEs incorporating non-hydrostatic pressure effects (Lu et al., 2015), the two-layer (depth-averaged) shallow-water equations (Cao et al., 2015), and the 3D Reynolds-averaged Navier–Stokes equations in σ coordinates (Ma et al., 2012).

In viewing of the main objective of employing a ZZ form of governing equation systems being to achieve the well-balanced property, and the fact that this property can also be satisfied by numerical schemes devised based on the conventional HH form (see, for example, Zhou et al., (2001); Hou et al. (2013); Hu et al. (2015)), some questions naturally arise. Is it really necessary to use a pre-balanced form of the equation system such as SWE-ZZ instead of SWE-HH, and what is the relation between numerical schemes designed based on different equation system forms? This study aims to address these issues. Note that in this study, we focus our attention on solving the SWEs using finite-volume method (FVM). In particular, we focus on state-of-the-art FVM schemes designed based on the approximate Riemann solvers.

The remainder of this paper is organized as follows. Section 2 presents the FVM discretizations for the SWEs, and revisits the well-balanced property for solving SWE-ZZ. In Section 3, a theoretical analysis is performed to explore the relation between numerical schemes solving the different forms of the SWEs. Finally, conclusions and discussions are presented in Section 4.

2. The well-balancedness of the HLLC-based numerical scheme for solving SWE-ZZ

In this section, we revisit the well-balanced property of SWE-ZZ. In the framework of FVMs, Eq. (1) may be explicitly discretized using the operator-splitting method as (Liang and Borthwick, 2009)

$$\begin{aligned}\boldsymbol{\varepsilon} = \begin{bmatrix}\varepsilon_c\\ \varepsilon_x\\ \varepsilon_y\end{bmatrix} &= \frac{\boldsymbol{U}_{i,j}^m - \boldsymbol{U}_{i,j}^k}{\Delta t}\\ &= \frac{\boldsymbol{F}_{i-1/2,j} - \boldsymbol{F}_{i+1/2,j}}{\Delta x} + \frac{\boldsymbol{G}_{i,j-1/2} - \boldsymbol{G}_{i,j+1/2}}{\Delta y} + (\boldsymbol{S}_0)_{i,j},\end{aligned} \tag{7a}$$

$$\frac{\boldsymbol{U}_{i,j}^{k+1} - \boldsymbol{U}_{i,j}^m}{\Delta t} = \boldsymbol{S}_f(\boldsymbol{U}_{i,j}^k, \boldsymbol{U}_{i,j}^{k+1}). \tag{7b}$$

Here, i and j denote the cell indexes; Δt and Δx denote the time step and space interval, respectively; "k", "m", and "$k+1$" denote the values at the old, intermediate, and new time levels, respectively; $\boldsymbol{\varepsilon}$ denotes the rate of change of the vector $\boldsymbol{U}$ at the intermediate time level; and $\boldsymbol{F}_{i\pm 1/2,j}$ and $\boldsymbol{G}_{i,j\pm 1/2}$ are the numerical flux vectors at the cell interfaces. The discretization of the friction force term (see Eq. (7b)) is irrelevant for this study, and the point implicit method (Fiedler and Ramirez, 2000) may be used to enhance numerical stability.

To calculate the numerical fluxes $\boldsymbol{F}_{i\pm 1/2,j}$ and $\boldsymbol{G}_{i,j\pm 1/2}$ in Eq. (7a), various approaches have been proposed in the literature. Here, we consider a widely used Godunov-type scheme, the HLLC (HLL with restoring the contact surface) approximate Riemann solver (Toro et al., 1994) for flux evaluations. Note that in the original proof of the well-balanced property of SWE-ZZ by Liang and Borthwick (2009), the HLL solver was used. The HLLC solver chosen here is a more sophisticated approach, which will be used in the analysis work in Section 3. Note that if an HLLC-based numerical scheme is well-balanced, then the corresponding HLL-based version should also be well-balanced. The HLL and HLLC solvers rely on evaluations of the eigenstructure of the equation system, and when the eigenstructure of the problem is known they can help resolve sharp fronts in the vicinity of discontinuities with high accuracy, and capture the wet/dry fronts accurately when applied for solving the SWEs (Toro, 2001). The HLLC solver estimates the