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A new well-posed vorticity divergence formulation of the shallow water equations

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ABSTRACT

A new vorticity–divergence formulation of the two-dimensional shallow water equations including boundary conditions is derived. The new formulation is necessary since the conventional one does not lead to a well-posed initial boundary value problem for limited-area modelling.

The new vorticity–divergence formulation includes four dependent variables instead of three and requires more equations and boundary conditions than the conventional formulation. On the other hand, it forms a hyperbolic set of equations with well-defined boundary conditions that leads to a well-posed problem with bounded energy.

1. Introduction

The vorticity–divergence form of the shallow water equations (SWE) is regularly used in global spectral modelling (e.g., Durran, 2010; Kantha and Clayson, 2000; Krishnamurti et al., 2006; Miller, 2007; Satoh, 2014). Oceanic applications using the vorticity–divergence form of the SWE are given by Frisius et al. (2009), Pearce and Esler (2010), Rajpoot et al. (2012) and Wang and Shi (2008). The SWE in a non-rotating frame are used in environmental and civil engineering applications (e.g., Sanders and Katopodes, 2000; Sanders, 2001; Sanders et al., 2008, 2010).

It has been demonstrated that using the vorticity and divergence as prognostic variables leads to advantages such as easy implementation of potential vorticity and potential enstrophy conservation principles and control of gravity waves via divergence damping. In addition, the vorticity and divergence are scalar variables in all coordinate systems (e.g., Durran, 2010; Ehrendorfer, 2012; Haltiner and Williams, 1980; Kantha and Clayson, 2000; Miller, 2007; Randall, 1994).

Nevertheless, excluding the spectral method, the vorticity and divergence variables are seldom employed in computational algorithms developed for global and limited-area models. The main reasons are (i) the difficulty in solving elliptic equations for the horizontal velocity components from the vorticity and divergence relations and (ii) the lack of suitable boundary conditions to close the system for limited-area domains. Efficient numerical algorithms such as the well-known multigrid technique (e.g., Trottenberg et al., 2001) or modern algorithms developed for solutions of linear systems (e.g., Boyd et al., 2013) can potentially be used to overcome the first drawback. In this paper we focus on the second drawback.

Well-posed boundary conditions for the SWE in terms of horizontal velocity components have been investigated by many researchers. For the one-dimensional SWE, well-posed boundary conditions have been derived by transforming them into a set of decoupled scalar equations (Durran, 2010; Miller, 2007). Oliger and Sundström (1978) derived well-posed boundary conditions for several sets of partial differential equations including the SWE by using the energy method. Ghader and Nordström (2014) derived a general form of well-posed open boundary conditions using similar techniques. The derivation of boundary conditions and well-posedness of the equations for ocean simulation models was the main subject of research in Palma and Matano (1998), Marchesiello et al. (2001), Treguier et al. (2001), and Blayo and Debreu (2005).

In this paper we derive a new vorticity–divergence formulation for the two-dimensional SWE including boundary conditions. The motivation for this is the realization that the conventional one does not lead to a well-posed initial boundary value problem for limited-area modelling. The core mathematical tool that we use is the energy method where one bounds the energy of the solution by choosing a

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minimal number of suitable boundary conditions (Gustafsson, 2008; Gustafsson et al., 1995; Nordström, 2007; Nordström and Svärd, 2005). In the initial stage of the analysis we also employ Fourier analysis.

The remainder of this paper is organized as follows. The three different forms of the SWE that will be discussed are given in Section 2. As an introduction to the limited-area problem, we briefly discuss the Cauchy problem in Section 3. In Section 4 we define well-posedness and show that the standard SWE in vorticity–divergence form does not lead to a well posed problem. The core content of the paper is given in Section 5 where we derive energy estimates and boundary conditions for the new SWE formulation, and show that it is well-posed. Finally, conclusions and future work are given in Section 6.

2. The shallow water equations

The inviscid single-layer shallow water equations (SWE), including the Coriolis term, are (Haidvogel and Beckmann, 1999; Miller, 2007; Pedlosky, 1987; Vallis, 2006)

$$\tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{v}\tilde{u}_y - f\tilde{v} + g\tilde{h}_x = 0 \tag{1}$$

$$\tilde{v}_t + \tilde{u}\tilde{v}_x + \tilde{v}\tilde{v}_y + f\tilde{u} + g\tilde{h}_y = 0 \tag{2}$$

$$\tilde{h}_t + \tilde{u}\tilde{h}_x + \tilde{v}\tilde{h}_y + \tilde{h}(\tilde{u}_x + \tilde{v}_y) = 0, \tag{3}$$

where $\tilde{u}$ and $\tilde{v}$ are the velocity components in the x and y directions respectively. Furthermore, $\tilde{h}$ represents the surface height, f is the Coriolis parameter, and g is the acceleration due to gravity.

Remark 1. The analysis in this paper is also valid for a varying Coriolis parameter since the skew-symmetry of the zero order terms are preserved. A change of coordinate system does not change the principle of the analysis, and an extension to spherical geometry is straightforward.

2.1. The linearized SWE in terms of velocities and height

The vector form of the two-dimensional SWE linearized around a constant basic state can be written

$$\tilde{\mathbf{u}}_t + \tilde{\mathbf{A}}\tilde{\mathbf{u}}_x + \tilde{\mathbf{B}}\tilde{\mathbf{u}}_y + \tilde{\mathbf{C}}\tilde{\mathbf{u}} = 0 \tag{4}$$

where $\tilde{\mathbf{u}} = (u, v, h)^T$ and the subscripts t, x, y denote the derivatives with respect to time and space respectively. The matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are

$$\tilde{\mathbf{A}} = \begin{pmatrix} U & 0 & g \\ 0 & U & 0 \\ H & 0 & U \end{pmatrix}, \quad \tilde{\mathbf{B}} = \begin{pmatrix} V & 0 & 0 \\ 0 & V & g \\ 0 & H & V \end{pmatrix}, \quad \tilde{\mathbf{C}} = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, u and v are the perturbation velocity components and h is the perturbation height. In addition, U, V and H represent the constant mean fluid velocity components and height.

2.2. The SWE in terms of vorticity–divergence and height

The vorticity $\tilde{\zeta} = \tilde{v}_x - \tilde{u}_y$ and divergence $\tilde{\delta} = \tilde{u}_x + \tilde{v}_y$ in Cartesian coordinates (x, y) can be used as prognostic variables instead of the two components of the velocity. By differentiating Eqs. (1) and (2) and combining them, the SWE in terms of vorticity, divergence and height become

$$\tilde{\zeta}_t + \tilde{u}\tilde{\zeta}_x + \tilde{v}\tilde{\zeta}_y + (\tilde{\zeta} + f)\tilde{\delta} = 0 \tag{5}$$

$$\tilde{\delta}_t + \tilde{u}\tilde{\delta}_x + \tilde{v}\tilde{\delta}_y + g(\tilde{h}_{xx} + \tilde{h}_{yy}) - f\tilde{\zeta} - 2J(\tilde{u}, \tilde{v}) + \tilde{\delta}^2 = 0 \tag{6}$$

$$\tilde{h}_t + \tilde{u}\tilde{h}_x + \tilde{v}\tilde{h}_y + \tilde{h}\tilde{\delta} = 0 \tag{7}$$

where $J(\tilde{u}, \tilde{v}) = \tilde{u}_x\tilde{v}_y - \tilde{u}_y\tilde{v}_x$.

The vector form of the two-dimensional SWE (5)–(7) linearized around a constant basic state can be written

$$\bar{\mathbf{u}}_t + \bar{\mathbf{A}}\bar{\mathbf{u}}_x + \bar{\mathbf{B}}\bar{\mathbf{u}}_y + \bar{\mathbf{C}}\bar{\mathbf{u}} + \bar{\mathbf{D}}\bar{\mathbf{u}}_{xx} + \bar{\mathbf{E}}\bar{\mathbf{u}}_{yy} = 0 \tag{8}$$

where $\bar{\mathbf{u}} = (\zeta, \delta, h)^T$ are the perturbation vorticity, divergence and height respectively. The matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$, $\bar{\mathbf{D}}$ and $\bar{\mathbf{E}}$ in (8) are

$$\bar{\mathbf{A}} = \begin{pmatrix} U & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & U \end{pmatrix}, \quad \bar{\mathbf{B}} = \begin{pmatrix} V & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & V \end{pmatrix}, \quad \bar{\mathbf{C}} = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & H & 0 \end{pmatrix}$$

$$\bar{\mathbf{D}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\mathbf{E}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 2. The differentiation of Eqs. (1) and (2) introduce the Laplacian of the height (as well as the divergence and vorticity as zero order terms). This removes the clean hyperbolic character of the formulation, which as we will show below, leads to significant stability problems.

2.3. The SWE in terms of vorticity–divergence and gradients of height

For reasons that will be explained in detail below, we introduce yet another form of the SWE, where we use the gradients of height as new variables. By differentiating also the height Eq. (3) with respect to x and y and combining the equations for the new variables $\tilde{h}_x, \tilde{h}_y$ with (5) and (6), we obtain the new extended system

$$\tilde{\zeta}_t + \tilde{u}\tilde{\zeta}_x + \tilde{v}\tilde{\zeta}_y + (\tilde{\zeta} + f)\tilde{\delta} = 0 \tag{9}$$

$$\tilde{\delta}_t + \tilde{u}\tilde{\delta}_x + \tilde{v}\tilde{\delta}_y + g((\tilde{h}_x)_x + (\tilde{h}_y)_y) - f\tilde{\zeta} - 2J(\tilde{u}, \tilde{v}) + \tilde{\zeta}^2 = 0 \tag{10}$$

$$(\tilde{h}_x)_t + \tilde{u}(\tilde{h}_x)_x + \tilde{v}(\tilde{h}_x)_y + \tilde{h}\tilde{\delta}_x + (\tilde{h}_x)\tilde{u}_x + (\tilde{h}_y)\tilde{v}_x + (\tilde{h}_x)\tilde{\delta} = 0 \tag{11}$$

$$(\tilde{h}_y)_t + \tilde{u}(\tilde{h}_y)_x + \tilde{v}(\tilde{h}_y)_y + \tilde{h}\tilde{\delta}_y + (\tilde{h}_x)\tilde{u}_y + (\tilde{h}_y)\tilde{v}_y + (\tilde{h}_y)\tilde{\delta} = 0. \tag{12}$$

The four variables $\tilde{\zeta}, \tilde{\delta}, \tilde{h}_x, \tilde{h}_y$ are determined by the four Eqs. (9)–(12).

Just as in the formulations above, we linearize Eqs. (9)–(12) around a constant state and obtain the vector form

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y + \mathbf{C}\mathbf{u} = 0 \tag{13}$$

where $\mathbf{u} = (\zeta, \delta, h_x, h_y)^T$ are the perturbation variables and $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are

$$\mathbf{A} = \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & U & g & 0 \\ 0 & H & U & 0 \\ 0 & 0 & 0 & U \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & V & 0 & g \\ 0 & 0 & V & 0 \\ 0 & H & 0 & V \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Remark 3. By introducing the gradients of the height $(\tilde{h}_x, \tilde{h}_y)$ as dependent variables, we remove the Laplacian in (6) $(\tilde{h}_{xx} + \tilde{h}_{yy} \to (\tilde{h}_x)_x + (\tilde{h}_y)_y)$, and reintroduce the hyperbolic character of the governing system. This will be shown below to restabilize the SWE which was destabilized by going from the velocity–height form to the vorticity–divergence–height form.

Remark 4. The linearization around a constant state performed above, is the most common and straightforward one. Linearization around variable states are more general, but often leads to excessive algebra and inconclusive results (since the sign of the gradients in the additional coefficient matrices are unknown). This is the case also for the formulations considered above. For more details on these matters, see Section 4.1 below.