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Approximations to the ocean's residual circulation in arbitrary tracer coordinates

Christopher L. Wolfe *

School of Marine and Atmospheric Sciences, Stony Brook University, Stony Brook, NY 11794-5000, United States

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ABSTRACT

The residual circulation is the flow which transports tracers. Its utility is tempered by the challenge associated with its computation: velocity must be mapped into tracer coordinates on a timescale which is short compared to eddy timescales. Several approximations have been introduced which allow the residual circulation to be evaluated using a small number of flow statistics, including the transformed Eulerian mean (TEM), the temporal residual mean (TRM), and the recently introduced statistical transformed Eulerian mean (STEM). This paper discusses the relationship between these approximations and illustrates their differences with a series of analytical and numerical examples. The STEM is found to be superior to the TEM and TRM in both its handling of the surface boundary condition and its ease of implementation.

1. Introduction

The residual circulation is a diagnostic for Lagrangian transport in the ocean. The precise definition of the "residual circulation" varies depending on the particular application. For oceanic tracer studies, perhaps the cleanest definition of the residual circulation is the thickness-weighted average (TWA) velocity in isotracer coordinates, as suggested by de Szoeke and Bennett (1993). Defined this way, the residual velocity has the appealing property that it is tangent to tracer isosurfaces if the flow is statistically steady and purely advective, i.e., without sources, sinks, or diffusion. There is, in principle, a distinct residual circulation for every tracer, but the most useful tracers are those directly related to the thermodynamics of the flow, such as heat or salt. When the tracer of choice is buoyancy, the resulting residual circulation summarizes the movement and transformation of water masses by diabatic processes.

Historically, there are two primary reasons to consider the residual circulation. In the first case, residual quantities are used to transform the equations of motion to isolate eddy terms so that they may be modeled or parameterized (as in, e.g., Andrews and McIntyre, 1976, 1978; Andrews, 1983; Treguier et al., 1997; Marshall and Radko, 2003, 2006; Gent et al., 1995; Radko, 2005, 2007; Ferreira and Marshall, 2006; Ito and Marshall, 2008). In the second case, the residual circulation is used as a diagnostic to understand the dynamics of ocean models (e.g., Henning and Vallis, 2004, 2005; Hallberg and Gnanadesikan, 2006; Hirabara et al., 2007; Cerovečki and Marshall, 2008; Cerovečki et al., 2009; Farneti et al., 2010; Mazloff et al., 2010; Wolfe and Cessi, 2010, 2011; Abernathey et al., 2011) and observations (e.g., Sloyan and Rintoul, 2001; Talley et al., 2003; Lumpkin and Speer, 2007; Iudicone et al., 2008; Macdonald et al., 2009; Zika et al., 2009, 2010). This paper is primarily concerned with the second case: the use of the residual circulation as a diagnostic tool.

While the residual circulation has become a popular diagnostic for studies of ocean dynamics, its calculation is a nontrivial undertaking in z-coordinate models: the velocity must be mapped into tracer coordinates on a timescale that is short compared to eddy timescales. This requires that the residual circulation either be calculated online (often contributing significantly to the cost of the simulation) or from numerous tracer and velocity snapshots (requiring large amounts of storage space). These problems are compounded when calculating the residual circulation from observations, which lack the temporal or spatial resolution to calculate the residual circulation directly. Approximate methods for calculating the residual circulation are a useful way of overcoming these challenges. This paper discusses and compares a number of techniques by which this may be accomplished and presents them using a consistent formalism which clarifies their connection to the TWA residual circulation.

The oldest of these techniques is the transformed Eulerian mean (TEM, Andrews and McIntyre, 1976), which was developed in the context of the atmosphere and requires only the mean horizontal velocity $\bar{\mathbf{u}}_H$, mean tracer $\bar{c}$, and the eddy correlation $\overline{\mathbf{u}'_H c'}$, where $\bar{\bullet}$

* Tel.: +1 6316323152.
E-mail address: christopher.wolfe@stonybrook.edu

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is an ensemble mean at constant height (typically approximated by a zonal mean or temporal low-pass filter). Noting that the TEM is more appropriate for zonally reentrant geometries such as the atmosphere than for enclosed oceanic basins, McDougall and McIntosh (1996, 2001) formulated the temporal (or three-dimensional) residual mean (TRM),[1] which extends the TEM to account for the transport of tracer variance $\overline{c'^2}$ by the mean flow. We will show that the TEM and TRM appear as formal expansions of the TWA residual circulation in the limit of small tracer variance and thus require that the tracer variance be small to give an accurate estimate of the TWA residual circulation. A new technique applicable to arbitrarily large tracer variance was recently developed in the atmospheric context by Pauluis et al. (2011, hereafter PSL). The key to this technique, termed the statistical transformed Eulerian mean (STEM), is to model the time dependence of the velocity and tracer fields as Gaussian random processes. This Gaussian approximation allows the residual circulation be estimated from the first- and second-order statistics of the flow.

The general properties of the residual circulation are reviewed in Section 2. Approximations to the residual circulation are discussed in Section 3, which summarizes the properties of the TEM, TRM, and STEM. These properties have been previously derived in the literature cited above, but it is helpful to have them set out in a consistent framework so that the various approximations can be more easily compared. We also develop expressions for the TEM and TRM streamfunctions in coordinates appropriate for arbitrary tracers which may not be monotonic in the vertical. Readers less interested in the formal aspects of the theory may wish to skip directly to Section 4, where the approximations to the residual circulation are illustrated using a series of examples ranging from simple kinematic models to output from eddy-resolving general circulation models. These examples show that the STEM proves to be the most accurate approximation to the residual streamfunction; in particular, the STEM handles boundaries with ease, while approximations based on series expansions tend to be poorly behaved near boundaries. Alternative definitions of the residual circulation are discussed in Section 5 and concluding statements are made in Section 6.

2. Definition of the residual circulation

The residual circulation is a device for separating, as much as possible, the advective transport of tracers from diffusive transport in unsteady flow. The most commonly used tracer is buoyancy b (proportional to density), but other tracers, such as salt and heat, can be useful (see, e.g., Ferrari and Ferreira, 2011; MacCready, 2011). In order to maintain generality, we develop the residual circulation in terms of a generic tracer c that is almost materially conserved, so that

$$c_t + uc_x + vc_y + wc_z = \varpi, \tag{1}$$

where ϖ represents the diabatic terms. For simplicity, we use Cartesian rather than spherical coordinates.

2.1. Tracer coordinates

The residual circulation is developed in tracer coordinates via the TWA formalism developed by de Szoeke and Bennett (1993), Smith (1999), Greatbatch and McDougall (2003) and Young (2012). We make use of the notation of Young (2012). In particular, we use coordinates (x, y, z, t) when working in height coordinates and $(\tilde{x}, \tilde{y}, \tilde{c}, \tilde{t})$ when working in tracer coordinates; partial derivatives with respect to the tilde coordinates are taken at constant c rather than constant z. See Appendix A for a summary of the results of Young (2012) which are pertinent to this paper.

Formally, the transformation into tracer coordinates can only be made if c is a monotonic function z and hence invertible (de Szoeke and Bennett, 1993). The TWA can, however, be generalized in a way that is valid for nonmonotonic tracer distributions (Nurser and Lee, 2004a). For any field $\theta(x, y, z, t)$, define its generalized TWA $\hat{\theta}(\tilde{x}, \tilde{y}, \tilde{c}, \tilde{t})$ to be

$$\hat{\theta}(\tilde{x}, \tilde{y}, \tilde{c}, \tilde{t}) \equiv \langle\sigma\rangle^{-1}\left\langle \int_{-H(x,y)}^{\eta(x,y,t)} \theta(x, y, z, t)\, \delta[c(x, y, z, t) - \tilde{c}]\, \mathrm{d}z \right\rangle, \tag{2}$$

where the mean differential thickness is

$$\langle\sigma\rangle(\tilde{x}, \tilde{y}, \tilde{c}, \tilde{t}) \equiv \left\langle \int_{-H(x,y)}^{\eta(x,y,t)} \delta[c(x, y, z, t) - \tilde{c}]\, \mathrm{d}z \right\rangle. \tag{3}$$

$H(x, y)$ is the depth of the ocean, $\eta(x, y, t)$ is the sea surface elevation, δ is the Dirac δ-function, and $\langle\bullet\rangle$ represents an ensemble average (typically approximated by a time mean or temporal low-pass filter) at constant $\tilde{c}$. We define $\hat{\theta} = 0$ whenever $\langle\sigma\rangle = 0$. The δ-functions in (2) and (3) have the effect of binning the rest of the integrand in tracer coordinates using infinitesimally thin bins. This generalized TWA has the same properties as the TWA formulated by Young (2012) and reduces to the TWA if $\tilde{c}$ is monotonic since the δ-functions are then simplified using the chain rule:

$$\delta[c(z) - \tilde{c}] = \frac{\delta(z - \zeta)}{c_z(\zeta)}. \tag{4}$$

Insertion of this result into (2) and (3) gives (A.1) and (A.2).

The mean isotracer height $\langle\zeta\rangle$ is found by integrating (3), giving

$$\langle\zeta\rangle(\tilde{x}, \tilde{y}, \tilde{c}, \tilde{t}) \equiv -\left\langle \int_{-H(x,y)}^{\eta(x,y,t)} \mathscr{H}[c(x, y, z, t) - \tilde{c}]\, \mathrm{d}z \right\rangle \tag{5}$$

where $\mathscr{H}$ is the Heaviside step function. The residual streamfunction is the mean volume transport above a given tracer isosurface (Nurser and Lee, 2004a):

$$\boldsymbol{\psi}(\tilde{x}, \tilde{y}, \tilde{c}, \tilde{t}) \equiv \left\langle \int_{-H(x,y)}^{\eta(x,y,t)} \mathbf{u}_H \mathscr{H}[c(x, y, z, t) - \tilde{c}]\, \mathrm{d}z \right\rangle. \tag{6}$$

The integral definitions (2), (5) and (6) ensure that the TWA, mean height, and residual circulation are properly defined even if c is not monotonic (as may happen if c is, e.g., salinity). If this is the case, then the integrals (5) and (6) will "fold" regions with $c_z < 0$ onto regions with $c_z > 0$ with the same tracer value (Nurser and Lee, 2004a). This folding effect must be kept in mind when interpreting plots of the residual circulation based on nonmonotonic tracers.

The TWA velocity $\mathbf{u}^\sharp$ (formally defined in Appendix A) is related to the streamfunction by

$$\langle\sigma\rangle\hat{\mathbf{u}}_H = -\boldsymbol{\psi}_{\tilde{c}} \quad \text{and} \quad \langle\zeta\rangle_{\tilde{t}} + \langle\sigma\rangle\hat{\varpi} = \nabla_c \cdot \boldsymbol{\psi}, \tag{7}$$

where

$$\hat{\mathbf{u}}_H \equiv \hat{u}\hat{\imath} + \hat{v}\hat{\jmath} \tag{8}$$

is the horizontal residual velocity and

$$\nabla_c \cdot \mathbf{v} \equiv \partial_{\tilde{x}}(\hat{\imath} \cdot \mathbf{v}) + \partial_{\tilde{y}}(\hat{\jmath} \cdot \mathbf{v}) \quad \text{for any } \mathbf{v} \tag{9}$$

is the horizontal "divergence"[2] at constant c. $\nabla_c \cdot \boldsymbol{\psi}$ is the diatracer component of $\mathbf{u}^\sharp$ scaled by $\langle\sigma\rangle$. No derivatives of the integral limits appear in (7) because all boundary terms vanish due to the

[1] Some authors (e.g., Greatbatch and McDougall, 2003; Jacobson and Aiki, 2006; McDougall et al., 2007) use the term TRM to refer to the full TWA rather an approximation to it. The reason for this appears to be that McDougall and McIntosh (2001) present an exact expression for the TWA streamfunction which is used to develop their approximate results. However, the use of the exact TWA in oceanography dates back at least to de Szoeke and Bennett (1993); the term TRM was first introduced in McDougall and McIntosh (1996), which was wholly concerned with approximate results. We thus prefer to reserve the label TRM for the approximate results.

[2] As pointed out by Young (2012), this is not a true divergence and does not satisfy a version of Gauss's theorem.