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Internal solitary waves shoaling onto a shelf: Comparisons of weakly-nonlinear and fully nonlinear models for hyperbolic-tangent stratifications

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ABSTRACT

In this study the evolution of internal solitary waves shoaling onto a shelf is considered. The results of high resolution two-dimensional numerical simulations of the incompressible Euler equations are compared with the predictions of several weakly-nonlinear shoaling models of the Korteweg–de Vries family including the Gardner equation and the cubic regularized long wave (or Benjamin–Bona–Mahoney) equation. Wave models in both physical x–t space and in s–x space are considered where s is a commonly used characteristic time variable. The effects of rotation, background currents and damping are ignored. The Boussinesq and rigid lid approximations are also used. The shoaling internal solitary waves generally fission into several waves. Reflected waves are negligible in the cases considered here. Several hyperbolic tangent stratifications are considered with and without a critical point. Among the equations in x–t space the cubic regularized long wave equation gives the best predictions. The Gardner equation in s–x space gives the best predictions of the shape of the leading waves on the shelf, but for many stratifications it predicts a propagation speed that is too large.

1. Introduction

Internal solitary-like waves (ISWs) are commonly observed in our oceans where they are usually generated by tide-topography interactions (Osborne and Burch, 1980; Pingree et al., 1986; Hibiya, 1988; Lamb, 1994; Gerkema, 2001; Zhao and Alford, 1989). They have been observed in many coastal sites, including the Scotian Shelf (Sandstrom and Elliott, 1984), the Australian North West Shelf (Holloway et al., 1997), the Bay of Biscay (Pingree et al., 1986), the Mauritanian Coast (Schafstall et al., 2010), the Sulu Sea (Apel et al., 1975), Strait of Gibraltar (Morozov et al., 2002), the Oregon Shelf (Moum et al., 2003; Moum et al., 2007) and East and South China Seas (Liu et al., 1998). Recent observations of shoaling ISWs on the New Jersey Shelf have shown examples of polarity reversal (Shroyer et al., 2009; Shroyer et al., 2010), mode-two solitary-like waves (Shroyer et al., 2010), and flat-crested waves (Shroyer et al., 2011). Remote sensing techniques often show a very complicated internal wave field with large numbers of overlapping, interacting wave packets generated at multiple sites (Hsu et al., 2000; Quaresma et al., 2007). Fig. 1 in Jackson et al. (2012) show the widespread occurrence of ISWs as observed from MODIS satellite imagery.

Exact solitary waves exist in an inviscid, horizontally homogeneous stratified fluid of constant depth with a rigid lid and in the absence of rotation (Turkington et al., 1991; Lamb, 2002). They do not exist in the ocean for several reasons, including the effects of rotation (Helfrich and Grimshaw, 2008), the free-surface, variable water depth, and horizontally varying stratification and currents. Never-the-less, solitary-like waves are commonly observed, the properties of which are often accurately predicted by solitary wave solutions of the full nonlinear equations and, indeed, by solitary wave solutions of a number of approximate nonlinear-dispersive wave equations (Ostrovsky and Grue, 2003; Vlasenko et al., 2005; Small and Hornby, 2005). In the ocean ISWs are typically hundreds of meters in length with amplitudes (maximum isopycnal displacement) on the order of tens of meters, although waves on the order of 200 m in amplitude and 2 km in length have been been observed in deep water, e.g., in the South China Sea (Klymak et al., 2006).

An early motivation for studying ISWs was the suspicion that they could be the source of large stresses suffered by off-shore oil-drilling rigs (Osborne and Burch, 1980). They are also important because they can transport fluid (Lamb, 1997) and energy large

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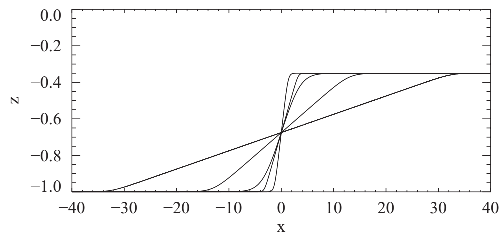

Fig. 1. Bathymetries used in the simulations. The bathymetric slopes are 0.01, 0.025, 0.1 and 0.25. For the steepest bathymetry and one of the two bathymetries with a slope of 0.1 $w = 0.5$. For the other bathymetries $w = 2.5$.

distances. As they propagate they can trigger instabilities in the bottom boundary layer beneath the waves (Bogucki et al., 2005; Carter et al., 2005; Moum et al., 2007; Stastna and Lamb, 2008; Aghsaee et al., 2012) and as they shoal into shallow water they can break (Helfrich, 1992; Michallet and Ivey, 1999; Bourgault and Kelley, 2007; Lamb and Nguyen, 2009; Vlasenko and Hutter, 2002). ISWs are believed to be at times an important source of vertical mixing in coastal oceans (Shroyer et al., 2010) and play an important role in nutrient dispersion and sediment transport (Huthnance, 1989). Because shoaling ISWs are a common occurrence it is important to understand the transformation of an ISW as it shoals.

Many laboratory experiments have been conducted to investigate the behaviour of a single ISW approaching a continental slope under conditions in which the pycnocline intersects the slope (Helfrich, 1992; Michallet and Ivey, 1999). Helfrich and Melville (1986) and Cheng et al. (2011) investigated ISWs shoaling onto a shelf using stratifications with the pycnocline lying above the shelf with the surface layer thicker than the lower layer on the shelf. Under these conditions shoaling waves pass through a critical point and undergo a polarity reversal. Experiments have also been conducted for cases in which the upper layer remains shallower than the lower layer on the shelf (Kao et al., 1985). The evolution of ISWs passing over a ridge has also been investigated (Sveen et al., 2002; Guo et al., 2004).

Numerical studies of shoaling ISWs generally fall into two categories: fully nonlinear models and weakly nonlinear models. Weakly-nonlinear models have been frequently used to interpret observations of shoaling waves and to explore their evolution (Small et al., 1999; Holloway et al., 1999; Grimshaw et al., 2004). These are typically based on the KdV equation and its extensions. An important extension that includes cubic nonlinearity is the Gardner equation, also called the extended KdV equation, which has the form

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \alpha_1\eta^2\eta_x + \beta\eta_{xxx} = 0, \tag{1}$$

where t is time, x is the horizontal coordinate, and $\eta(x,t)$ is the wave shape which for convenience can be interpreted as the maximum vertical isopycnal displacement. The coefficients are determined by the background stratification and currents and are given in terms of vertical structure functions (Lamb and Yan, 1996). Equations with higher-order dispersive terms have been derived (Lamb and Yan, 1996) and extended to a non-Boussinesq fluid with a free surface by Grimshaw et al. (2002). For shoaling waves the coefficients are functions of x and an additional shoaling term is added to the equation (next section). Additional terms to model the effects of damping or the Earth's rotation can also be added (Holloway et al., 1999; Grimshaw et al., 2004). Several mathematical and numerical studies of the KdV and Gardner equations with time-varying nonlinear coefficients have been carried out (Grimshaw et al., 1998; Nakoulima et al., 2004).

Other weakly-nonlinear models include the Joseph-Kubota equation for waves in a finite-depth fluid (Joseph, 1977; Kubota et al., 1978) and the Benjamin–Ono, or Benjamin–Davis–Ono, equation (BO) for stratifications with a thin upper layer (Benjamin, 1967; Ono, 1975). These are described in Liu et al. (1985) where the Joseph-Kubota equation is used to model waves in the Sulu Sea. Koop and Butler (1981) found that the KdV was more accurate in parameter regimes in which other equations would be expected to be more appropriate. We only consider equations of the KdV type here.

The Gardner equation is valid for small amplitude long waves. The linearized equation has the dispersion relation

$$\sigma = c_0 k - \beta k^3, \tag{2}$$

where k is the wave number and σ the wave frequency. Noting that $\beta > 0$ for internal waves, as $k \to \infty$ the frequency, phase speed $c = \sigma/k$ and group velocity $c_g = \sigma_k$ all go to $-\infty$ which is physically incorrect. It also leads to difficulties in finding numerical solutions of the KdV equation. For this reason many authors have studied the regularized long wave (RLW), or Benjamin–Bona–Mahoney (BBM), equation which is derived by using the dominant balance $\eta_t \approx -c_0\eta_x$, to replace one x derivative in the dispersive term by a time derivative. The RLW equation, extended to include the cubic nonlinear term, which we call the cubic RLW equation, is

$$\eta_t + c_0\eta_x + \alpha\eta\eta_x + \alpha_1\eta^2\eta_x - \frac{\beta}{c_0}\eta_{txx} = 0, \tag{3}$$

which has the linear dispersion relation

$$\sigma = \frac{c_0 k}{1 + \frac{\beta}{c_0}k^3}. \tag{4}$$

This has the physically realistic property that the phase speed and group velocity go to zero as $k \to \infty$ which simplifies finding numerical solutions. The phase speed goes to zero monotonically while the group velocity has a negative minimum of $-c_0/8$ at $k = \sqrt{3c_0/\beta}$ before going to 0 from below as $k \to \infty$. With full linear dispersion the phase speed and group velocity are always positive and decay to zero monotonically as $k \to \infty$ however their asymptotic behaviour depends on the stratification. For example if the buoyancy frequency N is constant $c \sim N/k$ and $c_g \sim Nm^2/k^3$ as $k \to \infty$ where m is the vertical wave number. For a two layer stratification c and c_g decay like $k^{-\frac{1}{2}}$ as $k \to \infty$. Shoaling weakly-nonlinear models based on the cubic RLW equation (3) have been used by Cai et al. (2002b) and Cai et al. (2002a).

Weakly-nonlinear models by their nature can not give a complete description of shoaling ISWs: they always have an amplitude limit; most are unidirectional, so wave reflection cannot be modelled (Boussinesq-type models are bi-directional); and most weakly-nonlinear models, including those considered here, are uni-modal and hence cannot describe the transfer of energy to other wave modes as an ISW shoals. To obtain a more complete description of shoaling ISWs, fully nonlinear numerical models have been used. For example, Bourgault and Kelley (2003) numerically solved the laterally-averaged Boussinesq Navier–Stokes equations to reproduce the experimental results reported by Michallet and Ivey (1999). The fully nonlinear numerical model used here has been used to investigate the formation of ISWs with trapped cores during shoaling (Lamb, 2002; Lamb, 2003) and to study shoaling waves in situations in which the pycnocline intersects the boundary (Lamb and Nguyen, 2009). Vlasenko and Hutter (2002) investigated ISWs shoaling onto a shelf, focussing on cases with steep slopes for which the shoaling waves broke. Shoaling over small shelf slopes was investigated by Vlasenko et al. (2005) who considered cases for which the waves shoaled adiabatically, i.e., cases for which the depth varied sufficiently