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On the stability and accuracy of the harmonic and biharmonic isoneutral mixing operators in ocean models

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ABSTRACT

Ocean models usually rely on a tracer mixing operator which diffuses along isoneutral directions. This requirement is imposed by the highly adiabatic nature of the oceanic interior, and a numerical simulation needs to respect these small levels of dianeutral mixing to maintain physically realistic results. For non-isopycnic models this is however non-trivial due to the non-alignment of the vertical coordinate isosurfaces with local isoneutral directions, rotated mixing operators must therefore be used. This paper considers the numerical solution of initial boundary value problems for the harmonic (Laplacian) and biharmonic rotated diffusion operators. We provide stability criteria associated with the conventional space–time discretizations of the isoneutral Laplacian operator currently in use in general circulation models. Furthermore, we propose and study possible alternatives to those schemes. A new way to handle the temporal discretization of the rotated biharmonic operator is also introduced. This scheme requires only the resolution of a simple one-dimensional tridiagonal system in the vertical direction to provide the same stability limit of the non-rotated operator. The performance of the various schemes in terms of stability and accuracy is illustrated by idealized numerical experiments of the diffusion of a passive tracer along isoneutral directions.

1. Introduction

Most ocean numerical models employ isoneutral[1] mixing operators either to parameterize the effect of unresolved mesoscale eddies (Gent and McWilliams, 1990; Smith and Gent, 2004), or more basically to control dispersive errors (Lemarié et al., 2012). It is thus very common for non-isopycnic models to implement a rotation of the diffusion tensor in a direction non-aligned with the computational grid. The benefits of a rotated mixing operator in simulating large scale flows are undeniable (e.g., Danabasoglu et al. (1994), Lengaigne et al. (2003)). Much of the improvements brought by the Gent and McWilliams (1990) parameterization of mesoscale eddies in coarse resolution models are also generally attributed to the orientation of lateral diffusive transport to be along isoneutral directions (Gent, 2011).

Redi (1982) provided the continuous form of the rotation tensor; however additional efforts were required to proceed to the actual implementation at the discrete level. Several works (Cox, 1987; Danabasoglu and McWilliams, 1995; Griffies et al., 1998; Mathieu et al., 1999; Beckers et al., 1998, 2000) tackled this problem that turned out to be more tedious than expected. The discretization in space raises difficulties to properly conserve the monotonicity (Mathieu and Deleersnijder, 1998; Beckers et al., 2000) and global tracer variance dissipation (Griffies et al., 1998) properties of the continuous operator once the problem is discretized. Moreover, due to the small vertical, relative to horizontal, grid distance typically used in numerical models the vertical and cross terms of the tensor can impose a severe restriction on the time step when explicit-in-time methods are used to advance the rotated operator. This stability problem is alleviated by the use of a standard backward Euler scheme for the vertical component of the tensor (Cox, 1987), at the expense of splitting errors[2] and associated errors in the balance between the active tracer isoneutral diffusive fluxes (Griffies, 2004, Chap. 16). This approach is used in all the state-of-the-art ocean climate models. The existing work on the isoneutral diffusion has been essentially carried out on the second-order (Laplacian) operator and under the small slope approximation (Cox, 1987; Gent and McWilliams, 1990).

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[1] Throughout the paper we use the terminology "isoneutral direction" (sometimes referred to as "epineutral direction" or simply "neutral direction" in the literature) which is tangent to the locally-referenced potential density surface and whose definition is purely local (McDougall, 1987), rather than "isopycnal direction" which more generally refers to the direction tangent to a potential density surface referenced to an arbitrary fixed pressure.

[2] In the context of this paper, splitting errors are associated with the splitting of the isoneutral Laplacian operator into a time-explicit part (the horizontal components and cross-derivative terms) and a time-implicit part (the vertical component).

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Although very few studied so far, a rotated biharmonic operator may be of interest for high resolution simulations due to its known property of scale selectivity. A biharmonic operator non-aligned with the direction of the computational grid is used in Marchesiello et al. (2009) and Lemarié et al. (2012), and discussed in Griffies (2004) (Chap. 14). Because global climate models are now targeting increasingly higher horizontal resolution, the question of the viability of such an operator is not only relevant for the regional modeling community but also for the ocean climate community. As an illustration, Hecht (2010) shows that for a 0.1^0 resolution global model the use of a Lax–Wendroff scheme with an intrinsic numerically-induced diffusion aligned with the horizontal direction leads to too much of a spurious dianeutral mixing in the Equatorial Pacific. This result suggests that even for eddy-resolving simulations an isoneutral mixing operator could be required. This motivates the design of a scale-selective (high-order) rotated operator. This is however not straightforward to maintain the stability of such an operator which can produce undesirable effects like overshooting/undershooting (Delhez and Deleersnijder, 2007) and spurious cabelling processes (Griffies, 2004, Chap. 14). To our knowledge, the current implementations of rotated biharmonic operators are based on an explicit Euler scheme in time with ad hoc tapering or clipping of the neutral slopes to maintain good stability properties (Marchesiello et al., 2009). This approach has however the undesirable effect to allow spurious dianeutral mixing even at places where the slopes are modest and satisfy the small slope approximation.

The aim of this paper is to study a set of space–time discretizations of the rotated harmonic and biharmonic mixing, and to assess them in terms of accuracy, stability and monotonicity violations. One additional constraint we impose to ourselves is accuracy relative to large grid slope ratios (defined as the ratio between the neutral slope and the aspect ratio of the computational grid) in order to make the scheme adequate for use in a terrain-following σ-coordinate model. Indeed, problems with σ models are generally more pernicious than with z-level models because it is not unusual that the slope between the computational grid and the isoneutral direction steepens to be greater than the grid aspect ratio. The paper is organized as follows. In Section 2 we introduce the formulation of the isoneutral mixing problem as well as three different ways to discretize the problem in space. Then Sections 3 and 4 are respectively dedicated to the temporal discretization of the rotated Laplacian and biharmonic operators. Section 5 provides the useful details to proceed to the actual implementation of the different schemes in ocean models. Finally, numerical experiments are designed to illustrate the properties of various space–time discretizations in Section 6. For clarity, the important notations used throughout the paper are given in Table 1.

An alternative approach to the use of isoneutral mixing operators is the design of a vertical coordinate system following the isopycnals (e.g., Hallberg and Adcroft, 2009; Hofmeister et al., 2010; Leclair and Madec, 2011). The present paper is a complementary effort in exploring the merits of different approaches to representing the nearly adiabatic flow in the oceanic interior.

2. Isoneutral mixing problem formulation

2.1. Continuous formulation

This section briefly introduces the continuous form of the problem under investigation throughout the paper. The three spatial directions are labeled x_1, x_2 for the horizontal coordinates, and x_3 for the vertical coordinate. We note q the tracer of interest, ∇

Table 1
Important notations for the three-dimensional analysis of the isoneutral mixing operators, where $m = 1,2,3$ denotes the three spatial direction.

State variables	
q	Three dimensional tracer (can be temperature or salinity)
ρ	Three dimensional density field
Coordinates and spatial operators	
x_1, x_2	Horizontal coordinates
x_3	Vertical coordinate, pointing upward.
Δx_m	Measure of the grid-box interface in the x_m direction
Δt	Time-step for the temporal discretization
$\partial_m = \partial_{x_m}$	Partial derivative in the x_m direction
$\mathscr{D}_2$	Isoneutral Laplacian operator, defined in (2.1)
$\mathscr{D}_4$	Isoneutral biharmonic operator, defined in (2.8)
$\mathbf{F} = (F^{(1)}, F^{(2)}, F^{(3)})$	Diffusive flux, defined in (2.7) at a continuous level and in (2.13), (2.14), and (2.19) at a discrete level
$\mathscr{J}_m(q,\rho)$	Jacobian determinant, defined as $\partial_m q\, \partial_3 \rho - \partial_m \rho\, \partial_3 q$
$\delta_m q$	Discrete differentiation in the x_m direction, defined in (2.12) (for $\delta_m \rho$ a particular instance of differentiation to allow computation of isoneutral directions is presented in Section 5.6)
Parameters	
κ_m	Diffusivity in the x_m direction
B_m	Hyperdiffusivity in the x_m direction
$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, 0)$	Neutral slope vector, defined in (2.5)
s_m	grid slope ratio, defined in (2.22)
β_m	Parameter controlling the stencil of the spatial discretization of isoneutral mixing operators, defined in (2.21)
$w^{\pm}$	Switches to select the computational stencil depending on the orientation of neutral slopes, defined in (2.15)
θ	Stabilizing parameter for the Method of Stabilizing Corrections, defined in (3.3)
$\tilde{\kappa}$	Stabilizing diffusivity for the Method of Stabilizing Corrections, defined in (4.1)
σ_m	Parabolic Courant number in the x_m direction, defined in (3.4)
$\sigma_m^{(4)}$	Square root of the biharmonic Courant number in the x_m direction, defined in (4.2)
z_{mn}	Discrete Fourier modes multiplied by Δt, defined in (3.14)
ϕ_m	Normalized Fourier frequency ($\lvert\phi_m\rvert \leqslant \pi$) in the x_m direction
λ	Exact amplification factor of the isoneutral Laplacian (Section 3) and biharmonic (Section 4) operators
$\tilde{\lambda}$	Approximate amplification factor obtained after space–time discretization of the isoneutral Laplacian (Section 3) and biharmonic (Section 4) operators
μ_2	Ratio between the maximum time-steps allowed for stability of the horizontal and the isoneutral Laplacian operators discretized using a forward Euler scheme, defined in (3.33)
μ_4	Same as μ_2 for the biharmonic operator, defined in (4.14)