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# Wave spectral moments and Stokes drift estimation

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ABSTRACT

The relationships between the moments of wave spectra and Stokes drift velocity are calculated for empirical spectral shapes and a third-generation wave model. From an assumed spectral shape and only an estimate of wave period and significant wave height, one may determine: the leading-order Stokes drift, other wave period estimates, and all spectral moments. The conversion factors are tabulated for quick reference for the common empirical spectral shapes. The different spectral shapes considered are shown to exhibit similar spectral moment relationships. Using these relationships, uncertainty in Stokes drift may be decomposed into the uncertainty in spectral shape and a much greater uncertainty due to significant wave height and wave period discrepancies among ERA40/WAM, satellite altimetry, and CORE2 reanalysis-forced WAVEWATCH III simulations. Furthermore, using ERA40 or CORE2 winds and assuming fully-developed waves results in discrepancies that are unable to explain the discrepancies in modeled Stokes drift; the assumption of fully-developed waves is likely the culprit.



## 1. Introduction

In situ observations, third-generation wave models, and satellites have begun to provide global estimates of the surface gravity wave field (e.g., Gulev et al., 2003; Caires et al., 2004; Ardhuin et al., 2009a,b; Collard et al., 2009; Hemer et al., 2010; Hanley et al., 2010). The Stokes drift velocity – the mean temporal and spatial difference between the Eulerian and Lagrangian velocities (hereafter Stokes drift) – is useful in calculating the transport of tracers (e.g., McWilliams and Restrepo, 1999) as well as the forcing of surface turbulence (e.g., Craik and Leibovich, 1976; Kantha and Clayson, 2004). However, accuracy and data coverage remain challenges in estimating wave properties, such as Stokes drift, globally. Indeed, McWilliams and Restrepo (1999) chose not to use ocean data in their pioneering global estimation of Stokes drift. At that time, using atmospheric data and the assumption of fully-developed waves (Pierson and Moskowitz, 1964) seemed more reliable than interpolations of buoy and ship data. However, recent climatologies of wave age reveal that assuming equilibration is not trustworthy (Hanley et al., 2010), as often wave state is dominated by developing or remotely-generated swell conditions.

This paper focuses on relationships useful for estimating Stokes drift from diverse ocean wave data sources, so comparisons may reveal persisting errors in data collection and modeling. The challenge in this endeavor is that data storage limitations, for example on buoys or in archived models, often result in loss of complete wave spectral information. Time series of just a few averaged quantities are typically retained, such as mean wave period and significant wave height. In addition, over the past few decades, the wave community has transitioned from an early preference for mean wave period (based on the first moment of frequency) to the zero-crossing wave period (based on the second moment), which provides improved statistical robustness (e.g., Gommenginger et al., 2003). To recover a simplified Stokes drift from these archived records and compare them, the connection between these different mean variables and Stokes drift needs specification.

For deep-water waves of limited steepness, the leading order Stokes drift for monochromatic waves at a specified depth depends inversely on the third power of wave period times the significant wave height squared (e.g., Phillips, 1966). Similarly, for unidirectional (but polychromatic) wave spectra at a specified depth, the Stokes drift depends on the improper integral of the power spectral density divided by the third power of wave period (Kenyon, 1969; McWilliams and Restrepo, 1999).

Dimensional analysis alone may provide a useful scaling for recovering Stokes drift for polychromatic waves with limited data. However, the precise relationship including numerical coefficients depends on the wave period estimate used, which is one indicator of the shape of the wave spectrum. It will be shown that based on knowledge of only a pair of spectral moments and an assumed spectral shape, the Stokes drift and other moments are readily

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estimated and these results are tabulated here. This procedure resembles finding relationships between the mean, variance, skewness, and kurtosis of standard probability distributions. For the wave problem, a spectral shape must be assumed to find these relationships; here well-known empirical spectra and one third-generation model simulation with arbitrary spectral shape (up to limited model resolution) are analyzed. The same calculations can be easily made for other spectra (e.g., Banner, 1990; Alves et al., 2003; Harcourt and D'Asaro, 2008), but are left to the reader or future investigator.

In this paper all Stokes drift approximations use the unidirectional wave assumption, common to other Stokes drift literature (Kenyon, 1969; McWilliams and Restrepo, 1999, etc.).[1] However in our third-generation model, it was found this assumption typically overestimates the leading order Stokes drift (with no wave field assumptions) by about 33% and is briefly addressed in Appendix A.4. Improving estimates of Stokes drift with multi-directional waves is an interesting geometrical problem from our perspective, as wave moments are typically based on scalar quantities, such as surface height variance, while the Stokes drift is a vector quantity. Numerical models easily handle this distinction, but the analysis here does not easily generalize. However, the goal here is an assessment of how well Stokes-related quantities are known and compare among data and models. For this purpose the unidirectional assumption gives a standard, physical, Stokes-related quantity that can be easily compared with limited data.

First this paper reminds the reader of spectral moment definitions and Stokes drift formulae, and then proceeds to evaluate relationships among these properties for different spectral shapes. The notation and conventions of McWilliams and Restrepo (1999) or Bouws (1998) are followed if possible, and other notation and a detailed presentation of Stokes drift is given in the Appendix. Finally, the uncertainty inherent in these approximations is compared to the discrepancies between different ocean wave data products. A forthcoming companion paper (Webb et al., in preparation) describes the climatology of Stokes drift and its relation to surface stress (i.e., the turbulent Langmuir number) over the eight-year window examined here, and the impact of regional variations in this climatology for surface mixing.

## 2. Spectral moments and observational definitions

It is common to summarize unidirectional or one-dimensional wave spectra at a point by their moments. The moments are defined by Bouws (1998) as

$$m_n = \int_0^\infty f^n \mathcal{S}_f(f)df, \tag{1}$$

where the (*wave*) *frequency spectral density*,[2] $\mathcal{S}_f$, is normalized to capture the variance of the surface height displacement, $\eta$, for some time scale $T$ such that[3]

$$\lim_{T\to\infty}\langle\eta(t)^2\rangle_T = \int_0^\infty \mathcal{S}_f(f)df. \tag{2}$$

Similarly, multidirectional or two-dimensional wave spectra can be summarized as

$$\widehat{m_n} = \int_0^\infty \int_{-\pi}^{\pi} f^n \mathcal{S}_{f\theta}(f,\theta)d\theta df, \tag{3}$$

where the *directional-frequency spectral density*, $\mathcal{S}_{f\theta}$, is normalized as

$$\lim_{T,L\to\infty}\langle\eta(\boldsymbol{x}_h,t)^2\rangle_{T,\boldsymbol{L}_h} = \int_0^\infty \int_{-\pi}^{\pi} \mathcal{S}_{f\theta}(f,\theta)d\theta df, \tag{4}$$

for some horizontal length scale $\boldsymbol{L}_h = (L,L)$.[4] By definition,

$$\int_{-\pi}^{\pi} \mathcal{S}_{f\theta}(f,\theta)d\theta \equiv \mathcal{S}_f(f). \tag{5}$$

In practice, wave spectra usually are calculated statistically using expected values for a particular frequency or deterministically as the limit of a finite sum over a limited area, such as a model grid point, as shown in Appendix A.2.2. Since wave amplitude decays exponentially with depth, we expect the 1D and 2D wave moments to decay in $z$ as

$$\lim_{T,L\to\infty}\langle\eta_{\tilde z}(\boldsymbol{x}_h,t)^2\rangle_{T,\boldsymbol{L}_h} = \lim_{T,L\to\infty}\frac{1}{TL^2}\int_{t-T/2}^{t+T/2}\int_{\boldsymbol{x}_h-\boldsymbol{L}_h/2}^{\boldsymbol{x}_h+\boldsymbol{L}_h/2}\eta_{\tilde z}(\boldsymbol{x}',t')^2 d\boldsymbol{x}'dt' \tag{6}$$

$$= \int_0^\infty \int_{-\pi}^{\pi} \mathcal{S}_{f\theta}(f,\theta)e^{\frac{8\pi^2 f^2}{g}z}d\theta df \tag{7}$$

$$= \int_0^\infty \mathcal{S}_f(f)e^{\frac{8\pi^2 f^2}{g}z}df. \tag{8}$$

The decay with depth depends on wavenumber $k$ or real frequency $f$, here related by the dispersion relation for linear deep-water waves ($4\pi^2 f^2 = gk$), where $g$ is the gravitational acceleration.

1D spectral moments are used to define traditional measures of wave properties clearly. The *spectral significant wave height*, $H_{m0}$, is a commonly used measure of wave height and is similar in magnitude to the *observed significant wave height*, $\overline{H}_{1/3}$.[5] It is defined as $H_{m0} = 4\sqrt{m_0}$ and typically ranges from $1.015\overline{H}_{1/3}$ to $1.08\overline{H}_{1/3}$ in wave observations (Ochi, 1998). Likewise, the ratios of moments, $T_n$, with dimensions in time given below, can be used to approximate the *mean wave period* $\overline{T}_m$ and *zero-crossing wave period* $\overline{T}_z$ (see Gommenginger et al., 2003)

$$T_n = \left(\frac{m_0}{m_n}\right)^{\frac{1}{n}}; \quad \overline{T}_m \approx \frac{m_0}{m_1}, \quad \overline{T}_z \approx \left(\frac{m_0}{m_2}\right)^{\frac{1}{2}}. \tag{9}$$

NOAA WAVEWATCH III (abbreviated here as WW3) commonly saves $T_{-1}$ (Tolman, 2009, p. 38). In calculating the surface Stokes drift, $T_3$ is ideal (see (13)). Often the spectrum is sharply peaked at a particular wave period, this period is known as the (*spectral*) *peak wave period*.

Significant wave height and one or more mean period estimates are often the only spectral data retained due to limited memory or estimated empirically (e.g., Gommenginger et al., 2003). Similar conventions apply to *mean wavelength*, where moments of the spectral distribution as a function of wavenumber are used.

With only limited spectral information, it may still be possible to estimate the Stokes drift accurately. The leading-order expression for the *full Stokes drift*, $\boldsymbol{u}^S$, from an arbitrary spectral shape is derived in the Appendix and given as

$$\boldsymbol{u}^S = \frac{16\pi^3}{g}\int_0^\infty \int_{-\pi}^{\pi}(\cos\theta,\sin\theta,0)f^3\mathcal{S}_{f\theta}(f,\theta)e^{\frac{8\pi^2 f^2}{g}z}d\theta df. \tag{10}$$

Notice that the horizontally-two-dimensional (henceforth $2D_h$) Stokes drift is a vector quantity whose magnitude depends both on the directional components of the wave field and the directional spread of wave energy (based on $\mathcal{S}_{f\theta}$) for each component. This can be quite complicated to estimate; however, a simpler unidirectional

[1] The unidirectional assumption supposes that there is a single wave direction for waves of all frequencies, so that wave direction can be neglected when calculations involving integration over frequency are performed. This assumption is *stronger* than the assumption of a typical wave direction with a spreading function about it (e.g., as in Donelan et al. (1985)). It will be required to go from (10) and (11).

[2] $f$ is ordinary (not angular) wave frequency.

[3] Angle brackets denote spatial or temporal averaging as indicated by the subscripts.

[4] The $h$ subscript denotes horizontal components.

[5] Hereafter, the *significant wave height* will refer to the *spectral significant wave height*, $H_{m0}$, unless otherwise specified.