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Accurate Boussinesq oceanic modeling with a practical, ‘‘Stiffened’’ Equation of State

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ABSTRACT

The Equation of State of seawater (EOS) relates *in situ* density to temperature, salinity and pressure. Most of the effort in the EOS-related literature is to ensure an accurate fit of density measurements under the conditions of different temperature, salinity, and pressure. *In situ* density is not of interest by itself in oceanic models, but rather plays the role of an intermediate variable linking temperature and salinity fields with the pressure-gradient force in the momentum equations, as well as providing various stability functions needed for parameterization of mixing processes. This shifts the role of EOS away from representation of *in situ* density toward accurate translation of temperature and salinity gradients into adiabatic derivatives of density.

In this study we propose and assess the accuracy of a simplified, computationally-efficient algorithm for EOS suitable for a free-surface, Boussinesq-approximation model. This EOS is optimized to address all the needs of the model: notably, computation of pressure gradient – it is compatible with the monotonized interpolation of density needed for the pressure gradient scheme in sigma-coordinates of Shchepetkin and McWilliams (2003), while more accurately representing the pressure dependency of the thermal expansion and saline contraction coefficients as well as the stability of stratification; it facilitates mixing parameterizations for both vertical and lateral (along neutral surfaces) mixing; and it leads to a simpler, more robust, numerically stable barotropic–baroclinic mode splitting without the need of excessive temporal filtering of fast mode. In doing so we also explore the implications of EOS compressibility for mode splitting in non-Boussinesq free-surface models with the intent to design a comparatively accurate algorithm applicable there.

1. Role of EOS in oceanic modeling

The Equation of State (EOS) relates the *in situ* density of seawater with its temperature (or potential temperature), salinity, and pressure (Θ, S, P, respectively),

$$\rho = \rho_{\mathrm{EOS}}(\Theta, S, P). \tag{1.1}$$

In a Boussinesq-approximation oceanic modeling code, *in situ* density does not appear by itself (since it is replaced by a constant reference density). Instead EOS and EOS-related quantities are needed for the following computations:

- Pressure-gradient force (PGF);
- Vertically averaged density $\bar{\rho}$ and the effective dynamic density for the barotropic mode ρ_* (vertically integrated pressure normalized by $gD^2/2$ where g is acceleration of gravity and D is the local column thickness), both of which participate in barotropic–baroclinic mode-splitting algorithm of Shchepetkin and McWilliams (2005,);
- Stability of stratification as well as external thermodynamic forcing (surface buoyancy flux) needed for mixing and planetary boundary layer parameterizations;
- Slope of neutral surfaces needed for horizontal (along-isopycnal) mixing (Griffies et al., 1998).

The purpose of this study is to review the present oceanic modeling practices for using EOS in these roles, focusing on the effects associated with seawater compressibility, and it assesses the consequences of the Boussinesq approximation in the context of a realistic EOS. The present study extends the analysis of consequences of Boussinesq approximation from Shchepetkin and McWilliams (2008).

This paper is organized as follows: Section 2 makes a overview of the Boussinesq approximation, analyzes the errors associated with using realistic seawater EOS, and introduces stiffening of EOS as a method to reduce these errors. Section 3 examines the consequences of finite compressibility of seawater for the four algorithmic roles outlined above, to establish requirements for

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the most suitable functional form of EOS. Special attention is given to barotropic–baroclinic mode-splitting since this subject is rarely discussed in the literature. Section 4 introduces a practical form of EOS and makes estimates of its accuracy. Section 5 explores the consequences of eliminating the Boussinesq approximation with the emphasis on accuracy of barotropic–baroclinic mode-splitting algorithm in the context of finite compressibility of seawater. Section 6 is the conclusion.

2. Boussinesq approximation and EOS stiffening

The Boussinesq approximation (*e.g.*, Spiegel and Veronis, 1960; Zeytounian, 2003, among others) replaces *in situ* density ρ with a constant reference value ρ_0 in all places where it plays the role of a measure of inertia, *i.e.*, everywhere except where multiplied by the acceleration of gravity g. The velocity field becomes non-divergent (incompressible) because the continuity equation changes its meaning from mass to volume conservation. This reverses the role of EOS from computing specific volume in a mass-conserving model to density ρ in a volume-conserving one. The conservation laws (momentum, energy, tracer content, etc.) are changed from mass-to volume-integrated; the thermodynamics is reduced to Lagrangian conservation (advection and diffusion) of Θ and S, while heating/cooling of fluid by adiabatic compression/expansion is neglected (except in the definition of potential temperature itself) and mechanical energy dissipated by viscosity is considered "lost" rather than converted into heat (Mihaljan, 1962). If EOS is a linear function of Θ and S, mass is conserved (along with volume) as a consequence of, and to the same degree as, the conservation of Θ and S; external heating/cooling produces a decrease/increase in mass, while keeping volume constant; and nonlinear EOS causes a Boussinesq model to conserve only volume, but no longer mass, even in the absence of external forcing.

The Boussinesq approximation brings simplifications by eliminating mass-weighting in a finite-volume code and limiting the role of EOS to the four purposes stated in Section 1. Except in the part of pressure-gradient term associated with perturbation of free surface in a free-surface model, ρ itself can be replaced with its perturbation, $\rho' = \rho - \rho_0$, because it appears only inside spatial derivatives (ultimately linked to the gradients of Θ and S) and only in the context of buoyancy commonly defined as $-g\rho'/\rho_0$, *i.e.*, normalized by ρ_0, whereas the equations can always rewritten in such a way that ρ_0 itself appears only in the context of this normalization and nowhere else. The Boussinesq approximation facilitates the barotropic–baroclinic mode-splitting in a pair of vertically-integrated free-surface and momentum equations, effectively uncoupling EOS from the barotropic mode. The incompressibility assumption eliminates acoustic waves regardless of whether the hydrostatic approximation is made.

However, physically important effects associated with seawater – cabbeling and thermobaricity – fundamentally require pressure-dependency in EOS and lead to the common practice of using the full non-approximated EOS, thus retaining its full compressibility even in an otherwise dynamically incompressible Boussinesq model. This obscures the concept of buoyancy because it can no longer be equated to the density (or potential density) anomaly, and can no longer be viewed as a Lagrangianly-conserved properly of the fluid, even thought Θ and S are. A related, frequently used approximation is the replacement of the full *in situ* pressure P in EOS (1.1) with its bulk hydrostatic reference value, $P \to -\rho_0 gz$, when computing ρ from the model prognostic variables, Θ and S, hence neglecting pressure variation due to baroclinic effects and essentially decoupling EOS pressure from the dynamic.

Under most offshore oceanic conditions ρ varies by $\approx \pm 3\%$ relative to its reference value; this leads to errors associated with the Boussinesq approximation which can be subdivided into two categories:

(i) errors relative to not using the Boussinesq approximation including not only quantitative, but conceptual as well, *i.e.*, excluded physical processes and/or missing/altered conservation laws; and
(ii) conflicts and internal inconsistencies caused the use of a realistic seawater EOS with full compressibility effects within a Boussinesq oceanic model.

Errors of type (i) are widely discussed in the literature (McDougall and Garrett, 1992; Dukowicz, 1997; Lu, 2001; Greatbatch et al., 2001; Huang and Jin, 2002; McDougall et al., 2002; Greatbatch and McDougall, 2003; Losch et al., 2004; Griffies, 2004; Young, 2010; Tailleux, 2009; Tailleux, 2010).

An example of an internal inconsistency of type (ii) can be illustrated by considering a barotropic compressible fluid layer whose density is a function of P alone (*i.e.*, because Θ, S are spatially uniform) and in hydrostatic balance,

$$\rho = \rho_{\rm EOS}(P) \quad \text{and} \quad \partial_z P = -g\rho. \tag{2.1}$$

Integration of (2.1) yields mutually consistent vertical profiles for P and ρ,

$$\begin{cases} P = \mathcal{P}(\zeta - z), \\ \rho = \mathcal{R}(\zeta - z) \equiv \rho_{\rm EOS}(\mathcal{P}(\zeta - z)), \end{cases} \quad \text{such that} \begin{cases} \mathcal{P}'(\zeta - z) = g\mathcal{R}(\zeta - z), \\ \mathcal{P}(0) = P|_{z=\zeta} = 0, \end{cases} \tag{2.2}$$

where $\mathcal{P}$ and $\mathcal{R}$ are universal functions of a single argument, *i.e.*, their structure depends only one the properties of $\rho_{\rm EOS}(P)$ in (2.1) but not directly on the local dynamical conditions, such as perturbation of the free-surface ζ. $\mathcal{P}'$ denotes derivative of $\mathcal{P}$ with respect to its argument (note that the sign is correct as stated above: both p and ρ increase with increase of $\zeta - z$, meaning increase downward). The condition $\mathcal{P}(0) = 0$ is the free-surface pressure boundary condition (for simplicity the atmospheric pressure is presumed to be constant and subtracted out).

A gradient of ζ induces a pressure gradient,

$$-\nabla_x P = -\mathcal{P}'(\zeta - z) \cdot \nabla_x \zeta, \tag{2.3}$$

and creates acceleration

$$-\frac{1}{\rho}\nabla_x P = -\frac{\mathcal{P}'(\zeta - z)}{\mathcal{R}(\zeta - z)} \cdot \nabla_x \zeta \equiv -g\nabla_x \zeta, \tag{2.4}$$

independently of vertical coordinate z regardless of the specific functional form $\rho_{\rm EOS}(P)$ in (2.1) and of the magnitude of the density variation within the column. Eq. (2.4) is derived without the use of Boussinesq approximation. Its Boussinesq analog is

$$-\frac{1}{\rho_0}\nabla_x P = -g\frac{\mathcal{R}(\zeta - z)}{\rho_0} \cdot \nabla_x \zeta. \tag{2.5}$$

The presence of the multiplier $\mathcal{R}/\rho_0$, which increases with depth, is clearly an artifact of the Boussinesq approximation. It results in an unphysical vertical shear in the acceleration, hence a spurious downward intensification of a geostrophically-balanced baroclinic current generated by a $\nabla_x \zeta$. Both these spurious effects are caused by the standard algorithmic chain in a Boussinesq model,

$$\Theta, S \to \rho = \rho_{\rm EOS}(\Theta, S, P = \rho_0 g(\zeta - z)) \to -\frac{1}{\rho_0}\nabla_x P. \tag{2.6}$$

To estimate the significance of this error, consider for simplicity a linear analog of (2.1),

$$\rho_{\rm EOS}(P) = \rho_1 + P/c^2, \tag{2.7}$$