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Accurate representation of geostrophic and hydrostatic balance in unstructured mesh finite element ocean modelling

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ABSTRACT

Accurate representation of geostrophic and hydrostatic balance is an essential requirement for numerical modelling of geophysical flows. Potentially, unstructured mesh numerical methods offer significant benefits over conventional structured meshes, including the ability to conform to arbitrary bounding topography in a natural manner and the ability to apply dynamic mesh adaptivity. However, there is a need to develop robust schemes with accurate representation of physical balance on arbitrary unstructured meshes. We discuss the origin of physical balance errors in a finite element discretisation of the Navier–Stokes equations using the fractional timestep pressure projection method. By considering the Helmholtz decomposition of forcing terms in the momentum equation, it is shown that the components of the buoyancy and Coriolis accelerations that project onto the non-divergent velocity tendency are the small residuals between two terms of comparable magnitude. Hence there is a potential for significant injection of imbalance by a numerical method that does not compute these residuals accurately. This observation is used to motivate a balanced pressure decomposition method whereby an additional "balanced pressure" field, associated with buoyancy and Coriolis accelerations, is solved for at increased accuracy and used to precondition the solution for the dynamical pressure. The utility of this approach is quantified in a fully non-linear system in exact geostrophic balance. The approach is further tested via quantitative comparison of unstructured mesh simulations of the thermally driven rotating annulus against laboratory data. Using a piecewise linear discretisation for velocity and pressure (a stabilised P_1P_1 discretisation), it is demonstrated that the balanced pressure decomposition method is required for a physically realistic representation of the system.

1. Introduction

Unstructured mesh methods offer the potential for simulations of a wider range of ocean phenomena than is possible using conventional structured meshes. The additional topological freedom offered by the approach enables the model mesh to conform to complex bounding topography, and for resolution to be varied locally so as to resolve flow regions of relatively increased dynamical importance. In addition, model resolution may be varied with time, via dynamic mesh adaptivity, to enable an optimised mesh to be maintained as flow features develop. Hence, while unstructured mesh methods may be more expensive for a given number of computational degrees of freedom, the extensibility of the methods allows simulations to be conducted in complex geometries, and for flow features to be resolved on multiple scales, at relatively decreased computational cost.

The geometric flexibility of unstructured mesh methods has previously been exploited to enable simulations in complex coastal geometries. Recent applications are described in Lambrechts et al. (2008), Wang et al. (2009) and Le Bars et al. (2010). In Chen et al. (2007) an unstructured finite volume model, the Finite Volume Coastal Ocean Model (FVCOM) is compared against two structured finite difference models in a number of idealised test cases. FVCOM is found to yield more accurate results, particularly as compared against finite difference solutions on grids with a piecewise-constant ("stair-cased") representation of the bounding topography.

On the global-scale, by far the majority of ocean models in use today are implemented using structured computational meshes (Griffies et al., 2000). A notable exception is the Finite Element Ocean Model (FEOM), which solves the hydrostatic primitive equations using tetrahedral meshes that are unstructured in the horizontal and aligned in the vertical (Danilov et al., 2004). FEOM has been applied to simulations of the North-Atlantic at eddy-permitting resolution (Danilov et al., 2005), and an extension including a sea-ice component has been applied to the global ocean (Timmermann et al., 2009).

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The strong asymmetry between the horizontal and vertical dimensions in the ocean typically leads one to choose structured, vertically aligned "chains" of nodes in the vertical, even where an unstructured mesh is chosen in the horizontal. However, a number of localised ocean phenomena are strongly non-hydrostatic and include significant vertical structure. These include open ocean deep convection (Jones and Marshall, 1993; Marshall et al., 1999) and gravity currents (see for example Legg et al. (2006) and Hiester et al. (2011) for recent vertically unstructured dynamic-mesh-adaptive simulations). In a multi-scale unstructured mesh ocean model there is the potential for such localised features, with significant vertical structure, to be resolved within a much larger, more coarsely resolved, system. Hence there is interest in the development of methods utilising more general vertical coordinates. The use of fully unstructured meshes in ocean modelling, with the relaxation of an imposed structure in the vertical dimension, is discussed in Pain et al. (2005) and Piggott et al. (2008a).

In this paper we consider the issue of physical balance representation on such general unstructured meshes. Geophysical systems, such as the atmosphere and ocean, are generally in geostrophic and hydrostatic balance to leading order. Since the dynamics are generally a small deviation from this balanced state, it is essential that any numerical model of geophysical systems is able to represent these balances accurately. Issues with representation of hydrostatic balance in σ-coordinate modelling are well-known (Gary, 1973; Mellor et al., 1994; Chu and Fan, 2003; Berntsen and Thiem, 2007). In Ford et al. (2004a,b) similar errors are addressed in unstructured mesh finite element modelling; pressure gradient errors for a flow over steep topography are identified and mitigated using an auxiliary solver for the hydrostatic pressure. This is a vertically unstructured analogue of the hydrostatic pressure solver described in Marshall et al. (1997a,b), albeit with no additional solver for a surface pressure component. In Piggott et al. (2008b) the approach is generalized via a decomposition of the full dynamical pressure into a "balanced pressure" component associated with buoyancy and Coriolis accelerations, and a "residual pressure" component.

Here we week to describe the mathematical basis for such pressure decomposition methods in the context of fully unstructured mesh finite element modelling. We further test the improvement in accuracy of balance representation for a number of finite element pairs. The method is found to lead to very significant improvements for certain discretisations (specifically the stabilised P_1P_1 finite element pair, detailed in Section 3.1), but very little for others. The balanced pressure decomposition method is therefore proposed as a method for allowing certain discretisations, and in particular low order stabilised finite element pairs, to be applied to unstructured mesh ocean modelling, without a prohibitive loss of physical balance accuracy.

In Section 2 we use a Helmholtz decomposition of the forcing terms in the momentum equation to motivate an auxiliary solver for the pressure associated with buoyancy and Coriolis accelerations (the "balanced pressure" component). In Section 3 a fully non-linear system in exact geostrophic balance is described, and used to quantify the increase in geostrophic balance accuracy when applying the balanced pressure decomposition method. In Section 4 a laboratory-scale geophysical analogue, the thermally driven rotating annulus, is used to validate the method against laboratory data; the computational cost of the method is also measured. Finally, Section 5 contains some concluding remarks, including a brief discussion of additional issues encountered when applying vertically unstructured mesh methods in large aspect ratio domains.

2. Formulation

In this section we motivate and formulate the balanced pressure decomposition method. In Section 2.1 the origin of geostrophic and hydrostatic balance errors is described via a Helmholtz decomposition of forcing terms in the momentum equation. In Section 2.2 it is shown that the fractional time-step pressure projection method may be modified, via the introduction of an additional diagnostic solve for the "balanced pressure" associated with the Coriolis and buoyancy accelerations, so as to increase the accuracy of physical balance representation. A solver for the balanced pressure is described in Section 2.3.

2.1. Continuous formulation

Consider the incompressible Navier–Stokes equations subject to Dirichlet boundary conditions:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{f} = -\nabla p + \boldsymbol{b} + \boldsymbol{F}, \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1b}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \text{ on } \partial\Omega, \tag{1c}$$

where $\boldsymbol{u}$ is the Eulerian velocity, $\boldsymbol{u}_D$ is the value of velocity on the boundary $\partial\Omega$ bounding the domain Ω, $\boldsymbol{f}$ is the Coriolis acceleration, $\boldsymbol{b}$ is the buoyancy acceleration, $\boldsymbol{F}$ contains all remaining forcing terms (including advection and any viscous dissipation) and p is the pressure (divided by a reference density ρ_0). We limit ourselves to consideration of Dirichlet boundary conditions satisfying the no-normal-flow condition $\boldsymbol{u_D} \cdot \hat{\boldsymbol{n}} = 0$ on $\partial\Omega$.

We make use of the Helmholtz decomposition (Weyl, 1940; Ladyzhenskaya, 1969; Denaro, 2003): for a vector field $\boldsymbol{G} \in L^2(\Omega)$ together with one boundary condition of:

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{G} = G_{\partial\Omega,n} \quad \text{on } \partial\Omega, \tag{2a}$$

$$\hat{\boldsymbol{n}} \times \boldsymbol{G} = G_{\partial\Omega,t} \quad \text{on } \partial\Omega, \tag{2b}$$

where $\hat{\boldsymbol{n}}$ is a unit normal on $\partial\Omega$, there exists a unique and orthogonal decomposition:

$$\boldsymbol{G} = \nabla\Phi + \nabla \times \boldsymbol{A} + \boldsymbol{H} \tag{3}$$

for some scalar potential $\Phi \in H^1(\Omega)$, vector potential $\boldsymbol{A} \in H^1(\Omega)$ and harmonic residual $\boldsymbol{H} \in L^2(\Omega)$. Note that while the terms in this decomposition are unique, the potentials themselves are not – one can add a constant to Φ and the gradient of a scalar to $\boldsymbol{A}$, with no influence on their gradient and curl respectively. $\nabla\Phi$ is curl-free (irrotational), $\nabla \times \boldsymbol{A}$ is divergence free (solenoidal) and $\boldsymbol{H}$ is both curl-free and divergence free. Hereafter the scalar potential gradient term $\nabla\Phi$ is referred to as the *conservative* component, denoted $\boldsymbol{G}_C$, and the remaining divergence free terms ($\nabla \times \boldsymbol{A} + \boldsymbol{H}$) are referred to as the *residual* component, denoted $\boldsymbol{G}_R$.

By incompressibility (1b), the Eulerian acceleration $\partial\boldsymbol{u}/\partial t$ is divergence free, and hence its conservative component is identically zero. Hence the conservative component in the Helmholtz decomposition of all forcing terms in the momentum equation, $(-\boldsymbol{f} + \boldsymbol{b} + \boldsymbol{F})$, is identically cancelled by the pressure gradient, and the scalar potential associated with the forcing terms in the momentum equation is identified as the dynamical pressure. This yields the familiar result that the dynamical pressure is slaved to the Eulerian acceleration, and acts only as a Lagrange multiplier via which the incompressibility constraint is applied (see, for example, Salmon (1988)). By Eqs. (1a) and (1b) and the no-normal-flow condition $\boldsymbol{u_D} \cdot \hat{\boldsymbol{n}} = 0$ on $\partial\Omega$, the pressure can be diagnosed up to a physically unimportant additive constant. The dynamics can be entirely described without consideration of the pressure, motivating the use of vorticity space formulations in which the pressure gradient is entirely absent.

We now perform separate Helmholtz decompositions of the Coriolis and buoyancy accelerations, $\boldsymbol{B} = -\boldsymbol{f} + \boldsymbol{b}$, and all other forcings, $\boldsymbol{F}$. Noting that the scalar potential component is associated with a pressure, this takes the form: