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Effects of rotation on self-resonant internal gravity waves in the ocean

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ABSTRACT

The *Resonant Triad Model* (RTM) developed in (Ibragimov, 2007), is used to study the *Thorpe's problem* (Thorpe, 1997) on the existence of *self-resonant* internal waves, i.e., the waves for which a resonant interaction occurs at second order between the incident and reflected internal waves off slopes. The RTM represents the extension of the *McComas and Bretherton's* three wave hydrostatic model (McComas and Bretherton, 1977) which ignores the effects of the earth's rotation to the case of the non-hydrostatic analytical model involving arbitrarily large number of rotating internal waves with frequencies spanning the range of possible frequencies, i.e., between the maximum of the buoyancy frequency (vertical motion) and a minimum of the inertial frequency (horizontal motion). The present analysis is based on classification of resonant interactions into the sum, middle and difference interaction classes. It is shown in this paper that there exists a certain value of latitude, which is classified as the *singular latitude*, at which the coalescence of the middle and difference interaction classes occurs. Such coalescence, which apparently had passed unnoticed before, can be used to study the Thorpe's problem on the existence of self-resonant waves. In particular, it is shown that the value of the bottom slope at which the second-order frequency and wavenumber components of the incident and reflected waves satisfy the internal wave dispersion relation can be approximated by two latitude-dependent parameters in the limiting case when latitude approaches its singular value. Since the existence of a such singular latitude is generic for resonant triad interactions, a question on application of the RTM to the modeling of enhanced mixing in the vicinity of ridges in the ocean arises.

1. Introduction

Recent observations based on measurement of microstructure show that mixing is considerably increased over bottom irregularities. For example, studies by Legg (2004), Legg and Adcroft (2003), Polzin et al. (1997), Wunsch and Ferrari (2004) conclude that one potential region for strong non-linear effects and mixing occurs over rough sloping bottom topography, e.g., Mid-Atlantic Ridge (see e.g., Thorpe, 2001 or Legg, 2004). In particular, the numerical results and analytical studies by Ibragimov (2008a) of internal tide generation over a continental slope confirm the (Lien and Gregg, 2001) predictions, based on measurements, that the turbulent dissipation is increased due to non-linear effects and mixing over ridges. The problem of reflection of internal waves from a plane boundary in different contexts was studied in Eriksen (1982), Ivey and Nokes (1989), Slinn and Riley (1998), Thorpe (2001), Thorpe (1997). Recent laboratory observations of reflected internal waves on sloping boundaries at a moderately large Reynolds number were reported in Gostiaux and Dauxois (2006). The particular question of reflecting internal wave beams was studied in Gerkema et al. (2006) and Tabaei et al. (2005). The group invariant properties of internal gravity wave beams considered in Tabaei et al. (2005) were also recently investigated in Ibragimov and Ibragimov (2009).

The problem which forms our main focus of interest here is to employ the *Resonant Triad Model* (RTM) developed in our previous studies in Ibragimov (2008a), Ibragimov (2007) to the *Thorpe's problem* in Thorpe (1997) on the interaction between internal waves reflecting from a sloping bottom inclined at a certain angle to the horizontal. The question of particular interest in this paper is "*Consider an internal wave whose energy propagates parallel to the group velocity in direction inclined at the given angle to the horizontal. Then, at what slope of a boundary are incident internal waves resonant, at second order, with the reflected waves?*"

This particular question has not been examined by Thorpe (1997), where the conditions for possible self-resonance for waves that are not in a plane normal to the slope were obtained. The results relevant to the present studies on reflection of internal waves from sloping topography, which will also be used in this paper, were summarized in LeBlond and Mysak (1978)). The discussion on the oceanic volume within which the incident and reflected waves composing a group overlap can interact has been reported recently in Thorpe (2001). The latter studies were motivated by

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the reason that, since the generation and interaction of waves is transient in the ocean, the waves can reach the boundary as short groups of finite dimensions (Thorpe, 2001). The results reported in Thorpe (1997), Thorpe (2001) and LeBlond and Mysak (1978)) were obtained with the ignored effects of the earth rotation. However, as it has been shown in the previous studies (see e.g., Ibragimov, 2008a, 2007 and MacKinnon and Winters, in press), the energy transfers within the oceanic internal wave field are strongly latitude dependent.

This article provides two independent answers to the above question: *direct* and *circumstantial*. Both answers are obtained from the RTM model. While the former answer is obtained as the direct application of the RTM model, the latter answer is the ingenious application of the latitude-dependent classification of resonant interactions. It is demonstrated on the example basis that, at a certain value of latitude, the RTM model (which assumes the boundary conditions at a flat bottom) can also be utilized to approximate the resonant interaction of internal waves impinging and reflected off a sloping boundary.

The model example provided in this article is based on studying two low modes resonant triads. The possibility of prolongation of the narrow modeling scenario studied here to the case of internal waves of low and high mode numbers limited by the size of the incident wave groups studied in Thorpe (2001) in the presence of the earth's rotation is a prospective direction for further studies.

2. Experimental model

Our starting point here is the weakly non-linear model of a two-dimensional stratified fluid motion away from frictional boundary layers in the simplest case of incompressible, uniformly stratified flow. Additionally, we focus on the deep ocean, so effects of barotropic advection are neglected in our studies. The simulations were done using a non-hydrostatic analytical model which describes two-dimensional flow vertically confined to lie between two horizontal rigid boundaries at $z = 0$ (rigid lid approximation for the free surface) and $z = -H$ at the flat bottom, where H is the range independent depth of the water column. Explicit viscosity and diffusion terms are ignored. Without loss of generality, it is assumed hereafter that the horizontal x coordinate is increasing eastward and the transverse horizontal coordinate y northward. It is also assumed that the horizontal length scales are smaller than the radius of the earth, so that the model is considered in a local Cartesian system on a tangent (x, z)-plane.

Under these flow conditions, within the Boussinesq approximation (the density variations are neglected everywhere except in the gravitational term), the governing two-dimensional equations for a stratified inviscid medium, observed in a system of coordinates rotating with angular velocity $\vec{\Omega}$, are

$$\rho_0\left[\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + 2\vec{\Omega}\times\vec{u}\right] = -\nabla p - g\rho\hat{k}, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\nabla\rho + w\frac{d\bar{\rho}}{dz} = 0, \quad (2)$$

$$\nabla\cdot\vec{u} = 0, \quad (3)$$

where ρ_0 is the constant reference density, ρ is the density perturbation from the ambient profile $\bar{\rho}(z)$, p is is the pressure deviation from the basic state, $\vec{u} = (u(x,z,t), v(x,z,t), w(x,z,t))$ is the velocity vector consisting of zonal, meridional and vertical components, $\hat{k}$ is the unit vector in the vertical (positive upward) direction z. The term $2\vec{\Omega}\times\vec{u}$ is referred to as the Coriolis acceleration.

Eqs. (1)–(3) are five non-linear equations for five unknowns $\vec{u}$, p and ρ'. Because of the *anisotropic* nature of the internal wave motion (the frequency of internal waves depends on the direction of the wavenumber vector $\vec{k}$ but not on its magnitude), the following dispersion relation between two-dimensional wavenumber vector, $\vec{k} = (k, m)$ and frequency of internal waves holds:

$$\omega^2 = N^2\sin^2\delta + f^2\cos^2\delta, \quad (4)$$

where $f = 2\Omega\sin\theta$ is the latitude-dependent Coriolis parameter, in which θ is latitude and $\Omega = 2\pi$ rad/day $\approx 0.73\times10^{-4}$ s^{-1} is the rate of the earth's rotation. Additionally, in the relation (4), N is a buoyancy frequency defined by

$$N = \left(\frac{-g}{\rho_0}\frac{d\bar{\rho}}{dz}\right)^{\frac{1}{2}} \quad (5)$$

and δ is the angle between the wavenumber vector $\vec{k}$ and the horizontal. As it is manifested from the dispersion relation written in the form (4), the frequency band for internal waves is limited, i.e., ω can change between the maximum of the buoyancy frequency N for the horizontal wavenumber vector (vertical motion) and the minimum of the inertial frequency f for the vertical wavenumber vector (horizontal motion). The analysis presented in this article is valid including the latter two limiting cases of internal waves propagation.

In present simulations, it is assumed that $N =$ const. While this simplification is commonly used in laboratory and theoretical studies and it is quite reasonable for the thermocline region, it is not common in the deep region of the ocean.

Here we summarize that part of the RTM which is essential for our analysis.

In two dimensions, the incompressibility condition (3) is taken into account to eliminate the pressure by introducing a stream function ψ via $u = \psi_z,\ w = -\psi_x$. To model *weakly non-linear interactions*, we look for solution of Eqs. (1)–(3) in the form

$$(\psi, v, \rho) = \sum_{i=1}^{\infty}\varepsilon^i(\psi_i, v_i, \rho_i), \quad (6)$$

where ε is a small parameter representing the ratio of typical particle speed to the typical phase speed of the waves. The appropriate bottom boundary conditions are $\psi(x,0,t) = \psi(x,H,t) = 0$, where the rigid lid approximation is assumed.

Due to the imposed boundary conditions, the *discrete vertical structure mode* eigenvalue problem for the stream function $\psi_1(x,z)$ is allowed. Namely, we can look for solutions of the $0(\varepsilon)$ problem in the following mode-discretized decomposition:

$$(\psi_1, \rho_1, v_1) = \sum_i a_i\left(\frac{\omega_i}{k_i}\sin\phi_i, \frac{N^2}{g}\sin\phi_i, \left[-f\frac{m_i}{k_i}\right]\cos\phi_i\right)\sin(m_i z), \quad (7)$$

where a_i is an amplitude of the isopycnal displacement, $\phi_i = k_i x - \omega_i t + \varphi_i$ is the phase of ith wave of the frequency ω_i and φ_i is a phase constant.

It is the well-known result of the previous studies outlined Section 1 that the weakly-non-linear interactions become evident at the second-order terms of the non-linear problem. Namely, the non-linear terms arising in the $0(\varepsilon^2)$ problem of Eqs. (1)–(3) give rise to sinusoidal waves with wave numbers and frequencies equal to the sum (or difference) of the wave numbers and frequencies of the primary waves, and proportional to the product of their amplitudes which means that the interaction of ith and jth wave produces a forcing term of the form $\cos(\phi_i \pm \phi_j)$ and $\sin(\phi_i \pm \phi_j)$ in the $0(\varepsilon^2)$ problem. Some of the terms may be solutions of the homogeneous problem. *Resonance* then occurs, and the forced waves may build up after a sufficient number of oscillations to be comparable in magnitude with the primary waves.

The resonant wave simulator reported in Ibragimov (2008a) represents the further development of the model reported in