Ocean Modelling 29 (2009) 15–26

Contents lists available at ScienceDirect

Ocean Modelling

journal homepage: www.elsevier.com/locate/ocemod

Reconciling estimates of the free surface height in Lagrangian vertical coordinate ocean models with mode-split time stepping

Robert Hallberg *, Alistair Adcroft

NOAA Geophysical Fluid Dynamics Laboratory, Ocean Group, 201 Forrestal Rd., Princeton, NJ 08540, USA
Atmospheric and Oceanic Sciences Program, Princeton University, Princeton, NJ 08540, USA

ARTICLE INFO

Article history:
Received 27 August 2008
Received in revised form 17 February 2009
Accepted 19 February 2009
Available online 3 March 2009

Keywords:
Ocean model
Split time stepping
Lagrangian vertical coordinate
Free surface

ABSTRACT

In ocean models that use a mode splitting algorithm for time-stepping the internal- and external-gravity modes, the external and internal solutions each can be used to provide an estimate of the free surface height evolution. In models with time-invariant vertical coordinate spacing, it is standard to force the internal solutions for the free surface height to agree with the external solution by specifying the appropriate vertically averaged velocities; because this is a linear problem, it is relatively straightforward. However, in Lagrangian vertical coordinate ocean models with potentially vanishing layers, nonlinear discretizations of the continuity equations must be used for each interior layer. This paper discusses the options for enforcing agreement between the internal and external estimates of the free surface height, along with the consequences of each choice, and suggests an optimal, essentially exact, approach.

1. Introduction

Hydrostatic ocean models filter out sound waves, so the fastest motions in such models are external gravity waves, propagating horizontally at $\sqrt{gH}$ (where g is the gravitational acceleration and H is the total ocean depth) – of order 200 ms^{-1} in the deep ocean. Shallow-water external gravity waves have nearly vertically uniform horizontal velocities and are well characterized by two-dimensional equations. The next fastest motions are horizontal velocities and internal gravity waves, both with speeds of a few meters per second and rich three-dimensional structures. Ocean models are about two orders of magnitude less costly to integrate in time if they separate integration of the external mode from the internal evolution of the model.

In models with time-invariant vertical coordinates (sigma- or Z-coordinate models or their stretched equivalent with a free surface), gravity waves are typically handled with an external mode solver (using either a rigid lid or a free surface). In either case, the time-filtered evolution of the free surface height gives a boundary condition on the vertical velocity, which is determined diagnostically from the vertically structured continuity equation. The discretization of the continuity equation in such models is invariably linear in the velocities, and it is straightforward to use a finite volume formulation and obtain exact consistency between the time-averaged external mode solution and the internal model structure. Even when the free surface height does vary modestly with time (such that no levels ever vanish with a given definition of the vertical coordinate), the algorithm used is still essentially the same; most importantly a linear (in velocity) discretization of the continuity equation is still appropriate, and a finite volume reconciliation of the changes in the interior structure with the evolution of the free surface (Griffies et al., 2001; Campin et al., 2004). With this exact finite volume reconciliation, there are no issues with tracer or mass (volume if Boussinesq) conservation.

By contrast, Lagrangian vertical coordinate models[1] use the continuity equation prognostically to describe the evolution of the thickness, h_k, of each vertically discrete layer k:

$$\frac{\partial h_k}{\partial t} = -\nabla \cdot (\boldsymbol{u}_k h_k). \tag{1}$$

(For simplicity, vertical fluxes and precipitation minus evaporation are ignored here – they do not alter the discussion.) The layer thicknesses can be summed vertically to obtain an estimate of the free surface height

$$\eta_h \equiv \sum_{k=1}^{K} h_k - D, \tag{2}$$

* Corresponding author. Address: NOAA Geophysical Fluid Dynamics Laboratory, Ocean Group, 201 Forrestal Rd., Princeton, NJ 08540, USA. Tel.: +1 609 452 6508; fax: +1 609 987 5063.
E-mail addresses: Robert.Hallberg@noaa.gov (R. Hallberg), Alistair.Adcroft@noaa.gov (A. Adcroft).

[1] See Adcroft and Hallberg (2006) for a full discussion of the differences between the solution approaches in Lagrangian and Eulerian vertical coordinate ocean models.

1463-5003/$ - see front matter Published by Elsevier Ltd.
doi:10.1016/j.ocemod.2009.02.008

where D is the time-invariant bottom depth.[2] The vertical sum of the layer continuity equations gives the barotropic continuity equation,

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot \left(\sum_{k=1}^{K} \boldsymbol{u}_k h_k \right) = -\nabla \cdot (\boldsymbol{U} H), \tag{3}$$

which uses the definitions of the total thickness and the barotropic velocity,

$$H \equiv \sum_{k=1}^{K} h_k \quad \text{and} \quad \boldsymbol{U} \equiv \frac{1}{H} \left(\sum_{k=1}^{K} \boldsymbol{u}_k h_k \right). \tag{4}$$

In the horizontally and temporally continuous equations, the two estimates of the free surface height, η and η_h, are perfectly consistent. However, discretization in time or in the horizontal spatial directions can break the consistency between Eqs. (1) and (3). A widely used approach to solve Eqs. (1) through (3) for η and η_h is the barotropic–baroclinic split time stepping scheme, in which the two-dimensional shallow water equations [(3) and the vertically averaged momentum equations] are used to estimate the evolution of the free surface height, η, for the time interval Δt, over which the full three-dimensional equations are also advanced. The two-dimensional equations can be advanced either explicitly with many short time steps (e.g. Killworth et al., 1991; Bleck and Smith, 1990; deSzoeke and Higdon, 1997; Hallberg, 1997; Shchepetkin and McWilliams, 2005) or implicitly (e.g. Dukowicz and Smith, 1994; Campin et al., 2004). Both the explicit and implicit approaches can be represented schematically as

$$\frac{\eta^{n+1} - \eta^n}{\Delta t} = -\nabla \cdot \langle \boldsymbol{U} H \rangle = -\nabla \cdot \langle \boldsymbol{V}(\boldsymbol{U}, H) \rangle, \tag{5}$$

where the angle brackets are used to denote whatever time averaging of velocities and thicknesses are used to determine the volume fluxes that advance the free surface height over a time step, Δt. The function $\boldsymbol{V}$ represents the spatial discretization of the barotropic fluxes, and may be a nonlinear function of $\boldsymbol{U}$ or H. The superscripts n and $n+1$ refer to successive (baroclinic) time levels (there may be many shorter sub-cycled time levels averaged over by the angle brackets). The precise meaning of the angle brackets is determined by the choice of split time stepping scheme. For many split explicit schemes (e.g., Bleck and Smith, 1990; Killworth et al., 1991; Hallberg, 1997; Higdon, 2005) the angle brackets are approximately a simple time average, while for Shchepetkin and McWilliams (2005) it would be a weighted filter that extends past time level $n+1$. With an implicit scheme, the angle brackets are likely to be the values at time level $n+1$.

The layer continuity equations are integrated over the same time period as (5) with a single large time-step, Δt; this is represented schematically as

$$\frac{h_k^{n+1} - h_k^n}{\Delta t} = -\nabla \cdot (\boldsymbol{u}_k h_k) = -\nabla \cdot \boldsymbol{F}(\boldsymbol{u}_k, h_k). \tag{6}$$

The layer thicknesses, h_k, must be non-negative, which generally requires the use in (6) of a discretization (here represented as the function $\boldsymbol{F}$) of the horizontal volume fluxes ($\boldsymbol{u}_k h_k$) that depends nonlinearly on the velocities, inevitably reverting to upwind differencing for sufficiently strong flow out of a relatively thin cell. For the discrete (in time and space) equations to have a consistent (single) estimate of the free surface height, the time average barotropic fluxes, $\langle \boldsymbol{V} \rangle$, and vertically integrated baroclinic fluxes, $\sum \boldsymbol{F}$, must satisfy

$$\langle \boldsymbol{V}(\boldsymbol{U}, H) \rangle = \sum_{k=1}^{K} \boldsymbol{F}(\boldsymbol{u}_k, h_k). \tag{7}$$

Failure to satisfy this constraint implies the existence of two estimates of the free surface and a possible inconsistency in the model equations. Satisfying this constraint is non-trivial due to the non-local in time and space nature of the constraint, and is especially non-trivial when either of $\boldsymbol{V}$ or $\boldsymbol{F}$ are non-linear.

The accumulated horizontal volume fluxes used to update the free surface height, $\langle \boldsymbol{U} \mathrm{H} \rangle$, can be related to an effective time-mean barotropic velocity by

$$\langle \boldsymbol{U} \rangle = \frac{\langle \boldsymbol{U} H \rangle}{\langle H \rangle} = \frac{\langle \boldsymbol{V}(\boldsymbol{U}, H) \rangle}{\langle H \rangle}, \tag{8}$$

where $\langle H \rangle$ is an appropriate time-mean total thickness. $\langle \boldsymbol{U} \rangle$ is commonly used in strategies to reconcile Eqs. (5) and (6).

At this point the layer equations could be advanced with a velocity whose thickness weighted vertical mean has been replaced by the time-mean barotropic velocity, $\langle \boldsymbol{U} \rangle$. Replacing the instantaneous vertically averaged velocity,

$$\boldsymbol{U}_0 \equiv \frac{\sum_{k=1}^{K} \tilde{h}_k \boldsymbol{u}_k}{\sum_{k=1}^{K} \tilde{h}_k}, \tag{9}$$

by the time-mean barotropic velocity gives new layer velocities of

$$\widehat{\boldsymbol{u}}_k = \boldsymbol{u}_k + \langle \boldsymbol{U} \rangle - \frac{\sum_{j=1}^{K} \tilde{h}_j \boldsymbol{u}_j}{\sum_{j=1}^{K} \tilde{h}_j} = \boldsymbol{u}_k + \langle \boldsymbol{U} \rangle - \boldsymbol{U}_0, \tag{10}$$

whose vertical average matches the time-mean barotropic velocity. Here the $\tilde{h}_k$ are estimates of the thicknesses at the faces of the control volumes, but except in the linear limit, $\tilde{h}_k$ cannot be guaranteed to agree with the effective thicknesses from the continuity equation, defined by $\widehat{h}_k \equiv \hat{\boldsymbol{n}} \cdot \boldsymbol{F}\left(\widehat{\boldsymbol{u}}_k, h_k\right) / \left(\hat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{u}}_k\right)$, where $\hat{\boldsymbol{n}}$ is the unit vector normal to the faces of the control volume. (In one-dimension, $\widehat{h}_k$ is just the volume flux divided by the velocity.) Eq. (10) alters the vertical mean velocity, but deviations from this mean

$$\boldsymbol{u}'_k = \boldsymbol{u}_k - \boldsymbol{U}_0, \tag{11}$$

are unaffected by this adjustment. With the adjustment in (10), an estimate of the layer time-filtered thicknesses can be advanced by

$$h_k^{n+1} = h_k^n - \Delta t \nabla \cdot \boldsymbol{F}\left(\widehat{\boldsymbol{u}}_k, h_k\right) = h_k^n - \Delta t \nabla \cdot \left(\widehat{\boldsymbol{u}}_k \widehat{h}_k\right). \tag{12}$$

Like the definition of the averaging in the angle brackets, the timing of the layer velocities in (10) and of the thicknesses used for the fluxes in (12) are determined by the underlying baroclinic time-stepping scheme. If a predictor-corrector scheme is used (as in the examples in Section 3), mass conservation only requires that the final correction to the layer thicknesses be consistent with (12), although it is often useful for the overall stability of the scheme if similar constraints to be applied to the predictor steps as well (Higdon, 2008).

In the limit where the volume fluxes vary linearly with the velocities, it is possible to select the discretizations of the fluxes and the thickness weights such that

$$\tilde{h}_k = \widehat{h}_k = \overline{h_k}^x \quad \text{and} \quad \langle H \rangle = \sum_{k=1}^{K} \overline{h_k}^x, \tag{13}$$

where the overbar-x represents the arithmetic mean of adjacent thicknesses (or any other plausible interpolation of the thicknesses to the velocity points that is independent of the velocity). So in the linear limit, summing (12) over the layers and subtracting the bottom depth, D, combined with (10) and (13) and the identity

[2] The discussion presented here makes the Boussinesq approximation. Without it the thicknesses would be measured in units of Pascals instead of meters, the roles of the bottom depth and sea surface height are replaced by surface and bottom pressure, and volume conservation becomes mass conservation. The discussion presented here would be identical without the Boussinesq approximation if this change of variables were made.