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A tale of two elements: $P_1^{NC}-P_1$ and RT_0

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ABSTRACT

The $P_1^{NC}-P_1$ and RT_0 finite element schemes are among the most promising low order elements for use in unstructured mesh marine and lake models. They are both free of spurious elevation modes, have good dispersive properties and have a relatively low computational cost. In this paper, we derive both finite element schemes in the same unified framework and discuss their respective qualities in terms of conservation, consistency, propagation factor and convergence rate. We also highlight the impact that the local variables placement can have on the model solution. The main conclusion that we can draw is that the choice between elements is highly application dependent. We suggest that the $P_1^{NC}-P_1$ element is better suited to purely hydrodynamical applications while the RT_0 element might perform better for hydrological applications that require scalar transport calculations.

1. Introduction

In the last 10 years, there has been an on-going effort to develop a new generation of marine models using unstructured rather than structured meshes. Several numerical methods have been investigated such as the finite element (FE), finite volume and spectral element methods. Among these three families of numerical methods, the FE method is the more general as the finite volume and spectral element methods can be seen as discontinuous and high order FE methods, respectively.

The late application of the finite element method to simulate marine flows is partly due to the issue of computational pressure modes, which were found to be present in most of the initial FE models and rendered them inaccurate (Walters and Carey, 1983; Walters, 1983; Walters and Carey, 1984). The approach originally proposed to avoid these modes was to use a modified form of the governing equation that does not support them. This method, called the wave equation method (Lynch and Gray, 1979), allows to use simple low order elements and accurately solves non-dispersive wave propagation problems. However, the wave equation formulation appears to be subject to advective instabilities and presents mass conservation issues (Kolar et al., 1994; Massey and Blain, 2006). The shortcomings of the wave equation formulation therefore lead to more research on finite element pairs to solve the primitive equations without having recourse to modified formulations or stabilization. Among the family of low order FE pairs, the $P_1^{NC}-P_1$ and RT_0 elements have appeared to have most of the desired qualities, i.e., absence of spurious modes, simplicity and good dispersive properties.

The lowest order Raviart–Thomas element (Raviart and Thomas, 1977), RT_0, tries to mimic the finite difference C-grid. Like the C-grid, the RT_0 element has spurious f-modes in the velocity but no spurious elevation modes (Raviart and Thomas, 1977; Hanert et al., 2003; Le Roux et al., 2007). However, there is usually no significant development of these modes so they are not an issue as long as the Rossby deformation radius is well resolved. The RT_0 FE scheme has been used in the unstructured mesh models developed by Walters and Casulli (1998) and Miglio et al. (1999). Other models based on a finite volume or finite difference formalism but using the same variables placement as the RT_0 element have also been developed (Casulli and Walters, 2000; Chen et al., 2003; Ham et al., 2005; Walters, 2005; Fringer et al., 2006; Stuhne and Peltier, 2006). The success of RT_0 is partly due to its formulation that has similarities with finite volumes although it is not a finite volume scheme.

The linear non-conforming, conforming element, $P_1^{NC}-P_1$, does not really have an equivalent Arakawa-type finite difference grid but has some similarities with the CD-grid of Adcroft et al. (1999) with the exception that the elevation lies on the vertices rather than at the center of the elements (Le Roux, 2005). The $P_1^{NC}-P_1$ has first been used by Hua and Thomasset (1984) to solve the shallow-water equations but then laid dormant for about 20 years before it was analysed by Le Roux (2005) and used by Hanert

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et al. (2005) to solve the non-linear shallow water equations. The $P_1^{NC} - P_1$ element has since then been used in unstructured mesh models developed by Greenberg et al. (2007), Sobolev et al. (2007), White et al. (2008a) and Lambrechts et al. (2008).

Although further developments should certainly be expected in the future, especially among fully discontinuous low and high order elements (Bernard et al., 2007; Giraldo and Warburton, 2008), here we present both FE pairs within an unified framework. This is the first time such a comparison has been made and it allows us to highlight their differences and similarities. In turn this allows us to make some recommendations regarding their domain of applicability. Given the widespread interest in this class of FE methods we believe this is of broad interest to not only the ocean modelling community but also to the limnology community as well. Both communities are actively developing these types of numerical solution techniques.

As usual for such comparative studies, we take the shallow water equations model as a benchmark problem. After having derived the discrete formulations for both elements in Section 2, we shall discuss their respective qualities by considering their propagation factors (Section 3), conservation properties (Section 4), convergence rates (Section 5) and the effects of different approximations (section 6). We conclude with recommendations concerning the types of problems for which they are best suited.

2. Formulation

The model equations are the two-dimensional shallow water equations. These equations are derived by vertically integrating the Reynolds-averaged Navier–Stokes equations and using the hydrostatic assumption and the Boussinesq approximation. The continuity and momentum equations are

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (H\mathbf{u}) = 0, \tag{1}$$

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{e}_z \times \mathbf{u} + g\nabla\eta + \mathscr{D} - \mathscr{T} = 0, \tag{2}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the depth-averaged horizontal velocity with components (u, v), f is the Coriolis parameter, $\mathbf{e}_z$ is the upward unit vector, g is the gravitational acceleration, $H = h + \eta$ is the total water depth, $h(\mathbf{x})$ is the water depth measured from a reference elevation, $\eta(\mathbf{x}, t)$ is the distance from the reference elevation to the free surface, $\mathscr{D}$ and $\mathscr{T}$ are dissipation and forcing terms, $\mathbf{x} = (x, y)$ is the horizontal coordinate, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is the material derivative and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the horizontal gradient.

No-normal flow boundary conditions are imposed on the boundary of the domain Ω ($\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ is the unit normal vector), which is assumed to be closed. Depending on the order of the dissipation term, some additional boundary conditions might be needed. These do generally not pose a problem and we will assume that the no-normal flow boundary condition is sufficient to find a unique solution. In the first part of the paper, we are going to use only the linearised shallow water equations, i.e., the elevation is neglected in front of the water depth in the continuity equation ($H = h$) and advection is neglected in the momentum equation. We shall only use the non-linear equations in Section 6.

2.1. Weak formulations

In order to obtain the finite element discretization of the linearised version of Eqs. (1) and (2), we first have to derive their weak formulation on the computational domain Ω. The latter is obtained by multiplying Eq. (1) and (2) by test functions $\hat{\eta}$ and $\hat{\mathbf{u}}$ and then integrating on Ω:

$$\int_\Omega \frac{\partial \eta}{\partial t}\hat{\eta}\, d\Omega + \int_\Omega \nabla \cdot (h\mathbf{u})\hat{\eta}\, d\Omega = 0, \tag{3}$$

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \hat{\mathbf{u}}\, d\Omega + \int_\Omega f(\mathbf{e}_z \times \mathbf{u}) \cdot \hat{\mathbf{u}}\, d\Omega + g\int_\Omega \nabla\eta \cdot \hat{\mathbf{u}}\, d\Omega + \int_\Omega (\mathscr{D} - \mathscr{T}) \cdot \hat{\mathbf{u}}\, d\Omega = 0, \tag{4}$$

$\forall \hat{\eta} \in \mathscr{H}$ and $\forall \hat{\mathbf{u}} \in \mathscr{U}$, where $\mathscr{H}$ and $\mathscr{U}$ are functional spaces defined later. In order to only have space derivatives of functions in $\mathscr{H}$ or in $\mathscr{U}$, we may integrate by parts either the divergence or the gradient term. Hence we respectively obtain the following weak formulations:

Find $\eta(\mathbf{x}, t) \in \mathcal{H}$ and $\mathbf{u}(\mathbf{x}, t) \in \mathcal{U}$ such that

$$\begin{aligned} &< \frac{\partial \eta}{\partial t}\hat{\eta} > - < h\mathbf{u} \cdot \nabla\hat{\eta} > + \ll \hat{\eta} h\mathbf{u} \cdot \mathbf{n} \gg = 0 \\ &< \frac{\partial \mathbf{u}}{\partial t} \cdot \hat{\mathbf{u}} > + < f(\mathbf{e}_z \times \mathbf{u}) \cdot \hat{\mathbf{u}} > + g < \nabla\eta \cdot \hat{\mathbf{u}} > \\ &\qquad + < (\mathcal{D} - \mathcal{T}) \cdot \hat{\mathbf{u}} > = 0 \\ &\qquad\qquad \forall \hat{\eta} \in \mathcal{H},\ \forall \hat{\mathbf{u}} \in \mathcal{U}, \end{aligned} \tag{5}$$

and

Find $\eta(\mathbf{x}, t) \in \mathcal{H}$ and $\mathbf{u}(\mathbf{x}, t) \in \mathcal{U}$ such that

$$\begin{aligned} &< \frac{\partial \eta}{\partial t}\hat{\eta} > + < \nabla \cdot (h\mathbf{u})\hat{\eta} > = 0 \\ &< \frac{\partial \mathbf{u}}{\partial t} \cdot \hat{\mathbf{u}} > + < f(\mathbf{e}_z \times \mathbf{u}) \cdot \hat{\mathbf{u}} > - g < \eta\nabla \cdot \hat{\mathbf{u}} > + g \ll \eta\hat{\mathbf{u}} \cdot \mathbf{n} \gg \\ &\qquad + < (\mathcal{D} - \mathcal{T}) \cdot \hat{\mathbf{u}} > = 0 \\ &\qquad\qquad \forall \hat{\eta} \in \mathcal{H},\ \forall \hat{\mathbf{u}} \in \mathcal{U}, \end{aligned} \tag{6}$$

where $\langle . \rangle = \int_\Omega . d\Omega$ and $\ll . \gg = \int_{\partial\Omega} . d\Gamma$. It should be noted that the impermeability boundary conditions can be naturally incorporated in formulation (5) by setting the boundary integral to zero. With that formulation, it is possible to select a functional space $\mathscr{U}$ containing only functions that either satisfy the boundary conditions or not. In other words, we can decide to impose the no-normal flow constraint only in a weak way (as typically natural boundary conditions are imposed in second order problems) or in the usual strong way thanks to an additional constraint on the functional space (Hanert and Legat, 2006).

Formulations (5) and (6) have been obtained from the model Eqs. (1) and (2) without making any assumptions about the numerical schemes that will be used to solve these equations. The solution to formulation (5) belongs to the functional spaces $\mathscr{H} = H^1(\Omega)$ and $\mathscr{U} = (L^2(\Omega))^2$ while the solution to formulation (6) belongs to $\mathscr{H} = L^2(\Omega)$ and $\mathscr{U} = H(\text{div}, \Omega) \equiv \{\mathbf{v} | \mathbf{v} \in (L^2(\Omega))^2 \text{ and } \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$.

2.2. Finite element discretizations

A finite element approximation to the exact solution of Eqs. (1) and (2) is found by replacing η and $\mathbf{u}$ by finite element approximations η^h and $\mathbf{u}^h$ in formulation (5) or (6). Those approximations respectively belong to finite dimensional spaces $\mathscr{H}^h \subset \mathscr{H}$ and $\mathscr{U}^h \subset \mathscr{U}$. They read

$$\eta \approx \eta^h = \sum_{i=1}^{M} \eta_i \phi_i,$$

$$\mathbf{u} \approx \mathbf{u}^h = \sum_{j=1}^{N} \mathbf{u}_j \psi_j \text{ or } \sum_{j=1}^{N} J_j \boldsymbol{\tau}_j,$$