Ocean Modelling 25 (2008) 144–153

Contents lists available at ScienceDirect

Ocean Modelling

journal homepage: www.elsevier.com/locate/ocemod

An approximated method for the solution of elliptic problems in thin domains: Application to nonlinear internal waves

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ARTICLE INFO

Article history:
Received 17 April 2008
Received in revised form 17 July 2008
Accepted 21 July 2008
Available online 30 July 2008

Keywords:
Nonhydrostatic approximation
Nonlinear internal waves
Dispersion relationship

ABSTRACT

Realistic numerical simulations of nonlinear internal waves (NLIWs) have been hampered by the need to use computationally expensive nonhydrostatic models. In this paper, we show that the solution to the elliptic problem arising from the incompressibility condition can be successfully approximated by a few terms (three at most) of an expansion in powers of the ratio (horizontal grid spacing)/(total depth). For an n dimensional problem, each term in the expansion is the sum of a function that satisfies a one-dimensional second-order ODE in the vertical direction plus, depending on the surface boundary condition, the solution to an $n-1$ dimension elliptic problem, an evident saving over having to solve the original n-dimensional elliptic problem. This approximation provides the physically correct amount of dispersion necessary to counteract the nonlinear steepening tendency of NLIWs. Experiments with different types of NLIWs validate the approach. Unlike other methods, no *ad hoc* artificial dispersion needs to be introduced.

1. Introduction

Nonlinear internal waves (NLIWs) have emerged in the past 30 years as a prominent feature of many shelf and coastal areas around the world. For a comprehensive catalog of observations, the reader is referred to Jackson (2004). In addition to being an interesting problem in itself, NLIWs impact several areas of coastal oceanography through enhanced mixing and transport (MacKinnon and Gregg, 2003; Leichter et al., 2003; Moum et al., 2003), biological oceanography by redistributing plankton (Pineda, 1999; Helfrich and Pineda, 2003; Scotti and Pineda, 2007), and geological oceanography by suspending and transporting sediments (Bogucki et al., 1997; Butman et al., 2006). For this reason, much research has been devoted to modeling NLIWs (Helfrich and Melville, 2006). Though reliable quantitative observations of NLIWs were available since the late 1960s (Ziegenbein, 1969; Halpern, 1971) they attracted serious consideration in the early 1980s (Osborne and Burch, 1980), when it was recognized that weakly nonlinear wave theory could be used to frame the problem (Benney, 1966; Liu and Benney, 1981). The assumption was that, properly normalized, the amplitude α and steepness β of these waves could be treated as small parameters in a series expansion. To the lowest nontrivial order this leads to the Korteweg–de Vries (KdV) equation (Korteweg and de Vries, 1895) for the amplitude of the waves.[1] The principal insight of KdV theory is that the steepening tendency of nonlinearity is balanced by the dispersive nature of the medium in which the waves propagate. This allows finite amplitude waves of special shape to propagate without distorsion. While attractive for its elegance and simplicity, KdV theory suffers from many shortcomings, which limit its usefulness as a predictive tool. It is well known that NLIWs in the ocean are often highly nonlinear, quite steep and have trapped cores (Stanton and Ostrovsky, 1998; Klymak and Moum, 2003; Scotti and Pineda, 2004); during generation and shoaling, topography couples modes, whereas KdV neglects mode-mode interaction and cannot handle steep topography; over long propagation times, rotation is important (Helfrich, 2007); three-dimensional effects can only be incorporated assuming weak dependence on the direction normal to the propagation; dissipation and instabilities require *ad hoc* treatment. While some of these concerns can be addressed within the KdV framework (Grimshaw and Smyth, 1986; Grimshaw et al., 1999; Smyth and Holloway, 1988; Holloway et al., 1997; Holloway et al., 1999), the fundamental limitations of the weakly nonlinear framework cannot be escaped. For this reason, newer models have been intro-

[1] In the two-layer approximation, the amplitude is the displacement of the interface separating the two layers; for a continuous stratification, waves are projected onto the normal modes of the linear problem, and KdV describes the evolution of the amplitude of a particular mode, in a frame of reference traveling with the linear phase speed of the mode.

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1463-5003/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ocemod.2008.07.005

duced, which typically remove the weakly-nonlinear constraint. The prototype is the (Choi and Camassa (1999)) model. This is a two-layer model, fully nonlinear but still dependent on the steepness of the waves to be small. It can be generalized to multiple layers, and can handle smooth topography, but it is not free from issues. For example, any amount of shear between layers, however small, will trigger Kelvin–Helmholtz instabilities that need to be filtered out. At the opposite end of the complexity scale we have the well established general circulation models (GCMs). Robust, well documented, with an extensive set of tools to address biological and geophysical problems, they have been used to address a wide range of problems. From an operational point of view, they would be the tool of choice to study NLIWs. Unfortunately, they have traditionally taken advantage of the hydrostatic approximation to reduce the computational load, and thus cannot provide the dispersion needed to counteract nonlinearity. Newer models, such as SUNTANS (Fringer et al., 2006) or the MITgcm model (Marshall et al., 1997) have been developed that do not make the hydrostatic approximation, and have been used to study NLIWs in realistic settings. However, despite the increase in computational power in the last decade, running these models for realistic problems is still extremely expensive. Not surprisingly, for these runs a significant fraction of the cost (as high as 60%, Fringer, personal comm.) is taken up by the solution of the three-dimensional elliptic problem associated to the incompressibility condition. This limits severely the resolution that can be achieved, and thus casts reasonable doubts on the predicted characteristics of NLIWs.

In this paper, we show that it is possible to relax the nonhydrostatic constraint so that a (suitably modified) nonhydrostatic model such as SUNTANS could be run to realistically simulate NLIWs without incurring the full cost of the nonhydrostatic case. The main objective is to introduce the appropriate amount dispersion in a controlled way, with tools that are available to a nonhydrostatic ocean model, while keeping the numerical overhead at a minimum. The method is based on a perturbative approach to the elliptic problem, inspired by how the theoretical models are derived. The crucial insight however is to recognize that the critical length scale upon which to base the expansion is not the physical length scales of NLIWs, but the numerical horizontal length scale of the grid. This idea may sound foreign to a mind accustomed to consider theoretical models as continuum objects which may eventually be solved numerically (if everything else fails). However, it is the natural way to approach the problem if we subscribe to the view that a GCM is a discrete numerical tool which tries to model a continuum system (the ocean).

Of course, it would be desirable to implement a similar strategy using a hydrostatic model as a starting point. As it will become clear during the foregoing discussion, this can be done rather easily if the hydrostatic model makes the rigid-lid approximation. If, however, the model uses a free-surface, application of the method described in this paper is not straightforward. Appendix A discusses some of the issues at stake. However, for the free-surface case none of them is satisfactory; a complete analysis is beyond the scope of this paper.

2. Analysis

The hydrostatic approximation is often introduced on dynamical grounds (see, e.g. Haidvogel and Beckmann, 1999, p. 21) whereas in fact, as will be shown below, it follows from a combination of kinematic and geometric constraints. Flows that occur on a horizontal scale L much larger than the local depth H are hydrostatic even in a microgravity environment. Conversely, it is wrong

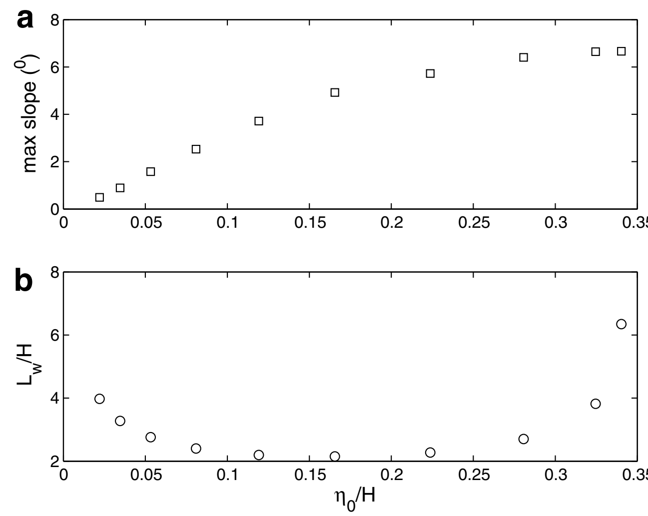

Fig. 1. Properties of solitary waves which are solutions of the DJL equation. (a) Maximum slope of isopycnals as a function of nondimensional wave amplitude. (b) Width of the waves.

to treat as hydrostatic flows with scales $L \simeq H$.[2] The issue with NLIWs is that their horizontal length scale is typically $O(H)$. Fig. 1 shows the width and maximum isopycnal slope of steady solitary waves generated with the Dubreil–Jacotin–Long (DJL) equation (Dubreil-Jacotin, 1937; Long, 1953) vs. wave amplitude η_0. To generate a wave, a solution of the DJL equation is obtained for a given total available potential energy (APE) (Scotti et al., 2006) using the technique described in Lamb and Wan (1998). For small APE values, the width L_w of the wave follows the KdV scaling $L_w = O(\eta_0^{-1/2})$. For the particular stratification considered here (standard hyperbolic tangent pycnocline), the width bottoms out at $\eta_0/H \sim .2$ (i.e. when amplitude matches the depth of the pycnocline), after which it begins to grow, as the solution approaches the conjugate state (Lamb and Wan, 1998). As the APE increases, the amplitude η_0 saturates as well, as can be seen from the flattening of the maximum slope curve. In other words, the depth of the pycnocline limits the amplitude of the waves. From a numerical point of view, these results suggest that an adequate horizontal resolution is $O(H/10)$, which is confirmed by numerical experiments (Scotti et al., 2007). On these scales, the hydrostatic approximation breaks down, and seems to imply that a correct simulation requires the use of a nonhydrostatic code. In the following, we show that this approach is overly conservative. It is possible to relax the nonhydrostatic condition while still having the correct dispersive behavior on the scales *relevant* to NLIWs propagation.

2.1. Solution method

Within an Euler solver, the incompressibility condition slaves the pressure to the instantaneous velocity and buoyancy field via an elliptic operator, the price paid for having filtered out the fast acoustical modes. Numerically, this means that we have to deal at some point with a Poisson problem, which for a generic curvilinear coordinate system, takes the form (Aris, 1989, p. 169–170)

$$\partial_i(Jg^{ij}\partial_j\phi) = \partial_i(Ju^{*i}). \tag{1}$$

The unknown ϕ can be the pressure or the potential in a projection scheme, but could also be the geopotential in a pressure coordinates

[2] This observation is of course not new, for example it is briefly hinted in Pedlosky (1986, p. 61).