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Semi-Lagrangian methods for a finite element coastal ocean model

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Abstract

Coastal ocean hydrodynamic models are subject to a number of stability constraints. The most important of these are the Courant–Friedrichs–Levy (CFL) constraint on gravity waves, a Courant (*Cr*) number constraint on advection, and a time step constraint on the vertical component of viscous stresses. The model described here removes these constraints using a semi-implicit approximation in time and a semi-Lagrangian approximation for advection. The accuracy and efficiency of semi-Lagrangian methods depends crucially on the methods used to calculate trajectories and interpolate at the foot of the trajectory. The focus of this paper is on evaluation of several new and old semi-Langrangian methods. In particular, we compare 3 methods to calculate trajectories (Runge–Kutta (RK2), analytical integration (AN), power-series expansion (PS)) and 3 methods to interpolate (local linear (LL), global linear (GL), global quadratic (GQ)) on unstructured grids. The AN and PS methods are both efficient and accurate, and the latter can be expanded in a straightforward manner to treat time-dependent velocity. The GQ interpolation method provides a major step in introducing efficient and accurate semi-Lagrangian methods to unstructured grids.

Keywords: Numerical model; Finite element; Semi-Lagrangian; Coastal ocean

1. Introduction

Coastal ocean models operate over a wide range of time and space scales, some that can be resolved and some that cannot. Typical time scales are of the order of the period of long waves (tens of minutes) and longer. Typical flow features range from tsunami propagation to seasonal baroclinic circulation. Embedded in these flows are short gravity waves which may or may not be of interest.

These models are also subject to several stability constraints. The most important of these are the Courant–Friedrichs–Levy (CFL) constraint on gravity waves, a Courant (*Cr*) number constraint on advection, and a time step constraint on the vertical component of horizontal viscous stresses.

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In the end, it is desirable to formulate the discrete equations so that there are no major stability constraints. Thus there would be no need for mode splitting to accommodate fast gravity waves as an example. The traditional way to eliminate time step constraints on gravity wave and viscous terms is to use a semi-implicit approach in time. Treating the advection terms is more problematic.

The stability of many explicit advection schemes is limited by *Cr*. Ideally, calculations that are stable at higher *Cr* but do not incur significant overhead are desired and Leonard (2002) shows how such schemes can be formulated. On the other hand, implicit schemes will involve the solution of a large matrix equation for the 3-dimensional velocity and thus have a large computational cost. One of the few stable methods available that does not require a matrix solution is a semi-Lagrangian approximation. This approximation is essentially explicit, removes the stability constraint, and does not necessarily involve a high computational cost.

Semi-Lagrangian methods take advantage of their simple formulation on a fixed grid and the inherent accuracy of integrating along streamlines. The equations are integrated in such a way that the trajectory ends at a fixed node at the end of each time step. Robert (1981, 1982) developed semi-implicit, semi-Lagrangian methods in a set of seminal papers and an extensive review of these methods for application to atmospheric problems is found in Staniforth and Côté (1991). Semi-Lagrangian schemes for the shallow water equations have been analysed in several papers (Casulli, 1987, 1990) where tracking methods, stability, artificial viscosity, and interpolation methods are considered. In an oceanic context, these methods have been used by Walters and Casulli (1998), Casulli and Walters (2000), Hanert et al. (2005), and Ham et al. (2005), among others.

Semi-Lagrangian methods seem ideal; however, the proper choice of tracking and interpolation methods is crucial for maintaining accuracy. Although high-order methods are well established for structured grids with regular quadrilaterals, they are difficult to implement on an unstructured grid (Staniforth and Côté, 1991; Le Roux et al., 1997). In a recent paper dealing with unstructured grids, Hanert et al. (2005) compared an explicit upwind scheme with a semi-Lagrangian scheme that uses a kriging interpolator. As an indication of efficiency, the semi-Lagrangian calculations were about ten times more expensive than the Eulerian calculations and both gave acceptable results for the large-scale test problems (Hanert et al., 2005). Hence, there is a need to increase the efficiency of these methods while maintaining accuracy.

The general problem is to solve $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{u}$ for a time step Δt to define the trajectory, then evaluate a function G at the foot of the trajectory. Because of the influence of coastal geometry, the streamlines tend to be curved. Hence, the methods of choice track the streamline rather than make a straight-line approximation such as used by Staniforth and Côté (1991) and Hanert et al. (2005). We consider three such methods in this paper.

Interpolating on an unstructured grid in an accurate and efficient manner is somewhat of a greater problem. Low-order methods tend to introduce considerable damping on the solution (McCalpin, 1988; Malcherek, 2001). On the other hand, high-order methods are difficult to implement and can be very inefficient (Hanert et al., 2005). One high-order method that requires high-order elements is presented in Xiu and Karniadakis (2001). We present here two low-order (linear) methods and compare these with a new high-order method.

In the following section, an overview of the numerical model is set forth with the details contained in the cited references. Next, the semi-Lagrangian methods are described in more detail including both tracking and interpolation. In Section 4, we present some comparative results for the different methods employed here. In Section 5, we summarise the conclusions.

2. Governing equations

The numerical model is formulated from the Reynolds-averaged Navier-Stokes equations (RANS) with a free surface. For the simulation of weakly dispersive surface waves, these equations can be averaged over the water depth to derive a set of equations similar to the standard shallow water equations but containing additional terms that describe non-hydrostatic forces (Walters, 2005). This procedure can be extended to solve the full Navier-Stokes equations.

However, for the purposes of this study a simplified set of equations are used to investigate different semi-Lagrangian advection approximations. In particular, the general equations are depth-averaged and the hydrostatic and Boussinesq approximations are used. The resulting equations are then a simple form of the shallow water equations.