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To continue or discontinue: Comparisons of continuous and discontinuous Galerkin formulations in a spectral element ocean model

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Abstract

The discontinuous Galerkin method is implemented in the spectral element ocean model to replace a continuous Galerkin discretization of the continuity and the tracer evolution equations. The aim is to improve the model's local conservation properties, and thus its performance in advection-dominated flows. The new model is validated against several oceanic benchmark problems, particularly ones that feature frontal structures and under-resolved features. Comparisons confirm the advantages of the DGM, including enhanced model robustness.

Keywords: Discontinuous Galerkin method; Spectral element method; Oceanic simulation; Shallow water equations; Ocean model

1. Introduction

The simulation of large scale geophysical flows raises a number of challenging computational problems associated with the representation of advection-dominated, rotating and stratified flows in thin fluid layers with steep topographic slopes (Willebrand and Haidvogel, 2001). The new generation of finite-difference-based ocean models have successfully addressed some of these issues which, along with the growth of available computational resources, has led to substantial improvements in performance. Most notable are the improvements to the models' advection schemes which are now locally conservative, upstream-biased (and generally third-order or higher) and often employ some form of limiting (Shchepetkin and McWilliams, 1998; Quartapelle, 1998; Warburton et al., 1998; Ezer et al., 2002).

The enforcement of the aforementioned properties in traditional finite element oceanic models is more complicated due to the unstructured nature of the grids and the Galerkin formulation. Upstream-biased

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finite-element schemes have commonly relied on stabilization methods (Brooks and Hughes, 1982; Hughes et al., 1989), but they incur a substantial computational cost, particularly for large simulations, as unsymmetric systems of algebraic equations must be solved at every time step. The Discontinuous Galerkin Method (DGM, see Cockburn, 1998; Warburton et al., 1999 and references therein for more details) offers a more natural setting in which to achieve the same aims while avoiding the inversion of big linear systems. DGM's advantages include: upstream-biased fluxes at element edges, enforcement of local conservation, and element-wise (independent) calculations of a discontinuously represented solution.

The success of DGM in simulating advection-dominated flows has prompted us to re-examine the solution algorithms within the spectral element ocean model (SEOM) (Iskandarani et al., 2003), particularly those concerned with the temperature and salt evolution equations. These equations are of the advection–diffusion type and are characterized by a very high Peclet number. The SEOM algorithms have thus far relied on the classical spectral element formulation; their behavior mimics that of high-order centered-difference schemes whereby unresolved frontal structures lead to numerical noise in the form of Gibbs oscillations, and to numerical instabilities. The C^0 continuity requirement is a further burden particularly in the presence of small-scale topography.

The present article focuses on assessing the benefits of DGM-based advection scheme for oceanic simulations. The emphasis is on improving the advection schemes currently used in SEOM, and on enhancing model robustness in under-resolved circumstances. Note that the DGM formulation adopted here can be described as a hybrid, since the momentum equations are still formulated using the traditional continuous Galerkin method (CGM). Oliger and Sundstrom (1978) and Browning and Kreiss (1986) show that the Riemann problem is ill-posed for inviscid hydrostatic primitive equations in a sence that it is impossible to find a unique set of characteristic directions on open boundaries. This leads to a difficulty in solving the Riemann problem on any element edge, making the full DGM approach problematic. Our formulation avoids this difficulty by applying the DGM to the pressure and tracer fields only. This formulation is also substantially cheaper than the full DGM, since the approximate Riemann solvers used in the latter are usually expensive. The DGM reformulation of the tracer evolution equations raises the issue of the proper treatment of the baroclinic pressure gradient term: the discontinuous tracers yield a discontinuous density and a discontinuous hydrostatic pressure. A similar consideration holds for the barotropic pressure if the sea surface height is also treated via DGM. A simple weak formulation of these pressure gradient terms is sufficient to evaluate these terms stably. Furthermore, spurious pressure mode are avoided by simply reducing the polynomial degree for the pressure by two as was done in Iskandarani et al. (1995).

The present article is structured as follows: Section 2 presents the CG and DG formulations for the shallow water equations. The new formulation is validated against several two-dimensional test problems in Section 3, Comparison between CGM and DGM is presented for a solution with a shock (Section 3.2); and for an unforced smooth solution with a planetary vorticity gradient (Section 3.1). Comparison of different strategies for stabilizing under-resolved simulations for both CGM and DGM is presented in Section 3.3. The new three-dimensional formulation is then presented in Section 4, and its performance is compared to that of the continuous Galerkin method in Section 4.1 for steep slope 3D simulation.

2. CGM and DGM for the shallow water equations

2.1. The shallow water equations (SWE)

The SWE are obtained by vertical integration of the three-dimensional Navier–Stokes equations along with the assumptions of hydrostatic pressure and a vertically uniform horizontal velocity profile. Let Ω be the two-dimensional region occupied by the fluid and let Γ denote its boundary. The reduced gravity SWE in Ω are given by the continuity and momentum equations:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot [h\mathbf{u}] = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{f} \times \mathbf{u} = \frac{\vec{\tau}}{\rho h} - g' \nabla \zeta - \gamma \mathbf{u} + \frac{\nabla \cdot [\nu h \nabla \mathbf{u}]}{h}, \tag{2}$$