Available online at www.sciencedirect.com
SCIENCE DIRECT®

Geoderma 134 (2006) 240 – 252

GEODERMA

www.elsevier.com/locate/geoderma

Multifractal Sierpinski carpets: Theory and application to upscaling effective saturated hydraulic conductivity

E. Perfect [a,*], R.W. Gentry [b], M.C. Sukop [c], J.E. Lawson [a]

[a] Department of Earth and Planetary Sciences, University of Tennessee, Knoxville, TN 37996-1410, USA
[b] Department of Civil and Environmental Engineering, University of Tennessee, Knoxville, TN 37996-2010, USA
[c] Department of Earth Sciences, Florida International University, Miami, Fl 33199, USA

Available online 19 April 2006

Abstract

Recent analyses of field data suggest that saturated hydraulic conductivity, K, distributions of rocks and soils are multifractal in nature. Most previous attempts at generating multifractal K fields for flow and transport simulations have focused on stochastic approaches. Geometrical multifractals, in contrast, are grid-based and thus better able to simulate distinct facies or horizons. We present a theoretical framework for generating two-dimensional geometrical multifractal K fields. Construction of monofractal Sierpinski carpets using the homogenous and heterogeneous algorithms is recalled. Averaging multiple, non-spatially randomized, heterogeneous Sierpinski carpet generators yields a new generator with variable mass fractions determined by the truncated binomial probability distribution. Repeated application of this generator onto itself results in a multiplicative cascade of mass fractions or multifractal. The generalized moments, $M_i(q)$, of these structures scale as $M_i(q)=(1/b^i)^{(q-1)Dq}$, where b is the scale factor, i is the iteration level and D_q is the q−th order generalized dimension, with q being any integer between $-\infty$ and ∞. This theoretical approach is applied to the problem of aquifer heterogeneity by equating the mass fractions with K. An approximate analytical expression is derived for the effective hydraulic conductivity, K_{eff}, of multifractal K fields, and K_{eff} is shown to increase as a function of increasing length scale in power law fashion, with an exponent determined by $D_{q\to\infty}$. Numerical simulations of flow in $b=3$, $D_{q\to\infty}=1.878$ and $i=1$ though 5 multifractal K fields produced similar increases in K_{eff} with increasing length scale. Extension of this approach to three dimensions appears to be relatively straightforward.

1. Introduction

How to describe, predict and simulate heterogeneity are pervasive issues in the fields of hydrogeology, petroleum engineering, and soil physics. Heterogeneities can occur in chemical and physical properties, both spatially and temporally. We are concerned with the spatial variation in physical properties, specifically the saturated hydraulic conductivity, K, of different geological facies or soil horizons. Such variations impact flow and transport in the subsurface, and thus have practical significance for the design and operation of pumping wells for human water use, oil production, and the spreading of contaminants in polluted soils and aquifers.

Increasingly, fractal-based models are being used to describe, predict and simulate aquifer heterogeneity (see for example the recent reviews by Neuman and Di Federico, 2003; Molz et al., 2004). Fractals are spatial or temporal patterns that repeat themselves at increasingly finer (or coarser) scales of resolution (Mandlebrot,

* Corresponding author. Tel.: +1 865 974 6017; fax: +1 865 974 2368.
E-mail address: eperfect@utk.edu (E. Perfect).

0016-7061/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.geoderma.2006.03.001

1982; Gouyet, 1996). The focus of this work is on two-dimensional spatial patterns constructed from a solid starting mass by an iterative process of mass removal and re-scaling.

As a concrete example, consider the Sierpinski carpet (named after the Polish mathematician Waclaw Sierpinski, 1882–1969) in Fig. 1. Construction based on the homogenous algorithm begins with a solid square of unit length (the *initiator*), which is divided into b^2 smaller squares of length $1/b$, where $b=2,3,4\ldots$ is a scale factor. At the first iteration level ($i=1$), n smaller squares are removed. In Fig. 1, $b=3$ and $n=1$. In subsequent iterations, this *generator* (Fig. 1A) is scaled down and applied to the remaining parts. In general, the number of remaining parts of length $1/b^i$ is given by $N(1/b^i)=(1/b^i)^{-D}$, where D is the mass fractal dimension defined as:

$$D \equiv \log(b^2-n)/\log(b) \quad (1)$$

For the example in Fig. 1, $D=1.892\ldots$ resulting in $N(1/3^1)=8$ for the first iteration, $N(1/3^2)=64$ for the second iteration and so on; the carpet produced after five iteration levels is shown in Fig. 1B.

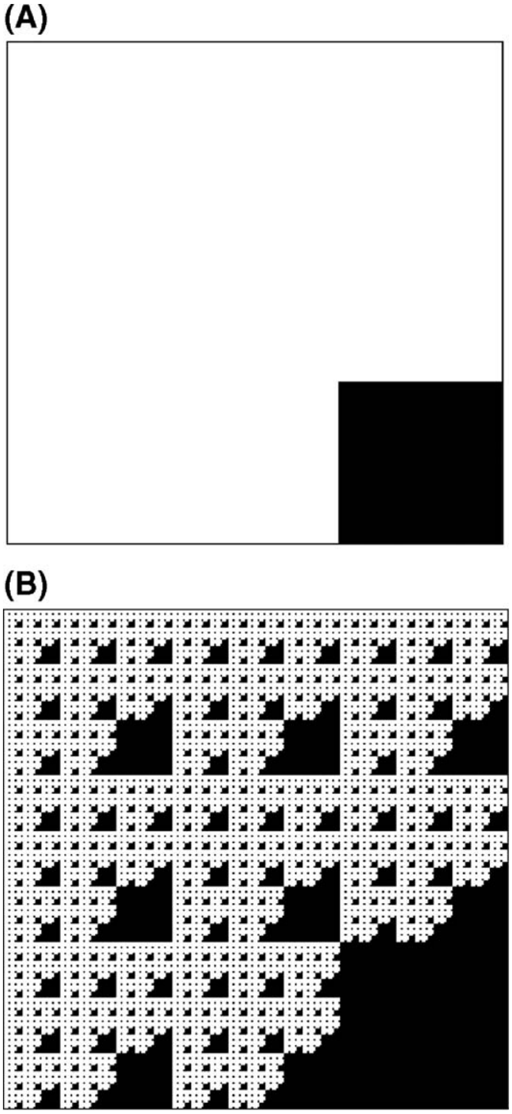

Fig. 1. Monofractal Sierpinski carpet with $p=8/9$ and $b=3$: (A) $i=1$ (generator) and (B) $i=5$. Remaining parts = white, removed parts = black.

The Sierpinski carpet and its three-dimensional cousin, the Menger sponge, have a long history of applications to natural porous media. They have primarily been used as models for pore spaces (Garrison et al., 1992, 1993) and fracture networks (Doughty and Karasaki, 2002) in rocks and soils. The percolation thresholds of randomized Sierpinski carpets were investigated by Sukop et al. (2002). In vadose zone applications, these fractals are often invoked in physically based derivations of the capillary pressure–saturation relation (Tyler and Wheatcraft, 1990; Bird et al., 1996; Perfect, 2005). Bird and Dexter (1997) and Sukop et al. (2001) studied the drainage characteristics of randomized Sierpinski carpets using a numerical invasion percolation algorithm. In an early application to aquifer heterogeneity, Wheatcraft et al. (1991) conducted numerical saturated flow and transport simulations in Sierpinski carpets; the carpets were used as a spatial model for facies with a bimodal K distribution.

More recently, detailed analyses of large data sets have revealed that K distributions of sedimentary rocks (Liu and Molz, 1997; Boufadel et al., 2000; Tennekoon et al., 2003) and soils (Giménez et al., 1999) are multifractal in nature. As will be explained in the next section, multifractals are characterized by a range of D values instead of a single fractal dimension, as is the case for the monofractal Sierpinski carpets discussed previously. Most attempts at generating multifractal K fields have concentrated on stochastic approaches (Boufadel et al., 2000; Tennekoon et al., 2003; Veneziano and Essiam, 2003). Numerical simulations of flow and transport in such fields have been reported by Veneziano and Essiam (2003, 2004). Numerical flow and transport simulations have also been performed in quasi-multifractal K fields generated with an algorithm based on bounded fractional Lévy motion (Painter and Mahinthakumar, 1999).

Compared to stochastic multifractals and Lévy motions, geometrical multifractals are grid-based and thus better able to simulate the spatial variability of K as a function of distinct geological facies or soil horizons.