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Horizontal solute transport from a pulse type source along temporally and spatially dependent flow: Analytical solution

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SUMMARY

Transport of solute mass transport, originating from a uniform pulse-type stationary point source through a heterogeneous semi-infinite horizontal medium, is studied. The heterogeneity is described by position dependent linear non-homogeneous expression for the velocity. The exponential unsteady variation in velocity of decreasing/increasing is also considered. The variation in dispersion parameter due to heterogeneity is considered proportional to square of that in the velocity. But the same due to unsteadiness is proportional to a power of the velocity which may take any value between 1 and 2 or outside this range. The variable coefficients of the two-dimensional advection–diffusion equation are put in degenerate form. These are reduced into constant coefficients with the help of new independent variables introduced at different stages, paving the way for using Laplace transformation technique to get the desired solution.

1. Introduction

The development of research in the field of solute transport through a medium is growing with the concern over the degradation of air, soil, surface water bodies and groundwater. Following the formulation of the advective–diffusion equation (ADE) and the theories relating dispersion and velocity (Taylor, 1953; Scheidegger, 1957; Rumer, 1962; Freeze and Cherry, 1979), the number of solute transport studies has increased considerably. Many of these studies aimed at solving the ADE for non-reactive and reactive solutes, subject to various initial and boundary conditions. A number of analytical solutions describing solute moving through one-dimensional media, considering adsorption, first-order decay and zero-order production, are compiled by van Genuchten and Alves (1982), Lindstrom and Boersma (1989). Many solute transport models concern homogeneous media, but in reality the ability of solute to permeate through the medium of air, soil or groundwater varies with position, which is referred to as heterogeneity. Early efforts to describe heterogeneity were achieved by making the use of stratification and defining porosity–distance relationship (Coats and Smith, 1964; Shamir and Harleman, 1967; Lin, 1977; Valocchi, 1989). Later scale-dependent dispersion and velocity have been attributed to heterogeneity.

Based on the observations of de Marsily (1986), analytical solutions to solute transport problem in a semi-infinite medium were obtained by Yates (1992), where the dispersion parameter depends on distance and increases up to some limited value. This problem was extended by Logan (1996) for periodic input condition and included the adsorption effects. A general methodology to develop dispersion models in three-dimensional heterogeneous aquifers under non-stationary conditions was presented by Serrano (1996). General solution for one-dimensional solute transport in heterogeneous porous media with scale-dependent dispersion was developed by Huang et al. (1996). Analytical solutions of the one, two, and three-dimensional ADEs were obtained by Hunt (1998). He assumed dispersivities that increase directly with the first power of the flow length for steady and unsteady flow. Hantush and Marino (1998) modeled depth-averaged solute transport and lateral-diffusive transport in a two-layer system of contrasting permeabilities. They obtained two-dimensional analytical solutions for the first-order rate model in an infinite medium, using the methods of Fourier and Laplace transforms. Analytical solution was obtained to analyze the effects of space dependent reaction coefficients on the one-dimensional transport of solute through soil by Flury et al. (1998). Zoppou and Knight (1999) provided analytical solutions for two- and three-dimensional ADEs by assuming a velocity component proportional to distance variable and corresponding diffusion coefficient proportional to square of the velocity. Quadruple method is implemented in order to simulate the effects of heterogeneities on one-dimensional advective and diffusive transport of a passive solute in porous media by Didierjean et al. (2004). Sander and Braddock (2005) and Su et al. (2005) considered dispersivity in the form of separable power-law

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dependence on both time and scale, and derived explicit closed form solutions for describing solute transport in saturated heterogeneous porous media. The limitations of analytical solution for ADE with coefficients being function of space variable have also been analyzed by Neelz (2006). A numerical model to simulate transient flow and solute transport through a variably saturated zone by using the mixed form of Richards equation for the solution of the flow component (being the mass conserving) and ADE for the solute transport was presented by Soraganvi and Mohan Kumar (2009).

Transport process in heterogeneous geological media is being modeled in recent works by fractional advection–diffusion equation (FADE) (Huang et al., 2008; Kim and Kavvas, 2006; Du et al., 2010). But such mathematical models can be solved numerically only. As FADE is the generalization of classical ADE hence the latter's analytical solutions in different real cases are essential to validate the numerical solution of a model comprising of the former, and those of more comprehensive models using ADE. It is the reason why the analytical solutions of advective–diffusive transport problems continue to be of interest in many areas. The literature presents several methods to analytically solve the ADE (Guerrero et al., 2009). Exact solutions of linear diffusion problems by classical integral transform techniques were reviewed and classified by Mikhailov and Ozisik (1984). They identified and unified seven classes of problems and demonstrated many applications in heat and mass diffusion. This work was generalized and extended by Kota (1993), thereby creating a new systematic procedure referred to as the Generalized Integral Transform Technique (GITT) used in recent works (Moreira et al., 2009; Cassol et al., 2009; Guerrero and Skaggs, 2010).

The objective of the present study is to get the analytical solution of a two-dimensional advection–diffusion equation with variable coefficients. The variability of the coefficients (dispersivity of the solute transport and velocity of the flow domain) is considered in more general and reasonable form. The medium is considered a semi-infinite heterogeneous horizontal domain. The heterogeneity is assumed of linearly increasing nature but of small order. Hence the velocity is linearly interpolated as a non-homogeneous function of increasing nature in position variable in a finite domain (in which concentration values are to be evaluated). Another variation in velocity is also assumed due to unsteadiness of exponential nature. According to Freeze and Cherry (1979), the dispersion parameter is proportional to a power, n of the velocity which ranges between 1 and 2. According to earlier theories cited at the outset, n is either 1 or 2. In the present analysis, due to heterogeneity, n is considered 2 but due to the unsteadiness of exponential nature, it may have any value between 1 and 2 or outside this range. Like (Sander and Braddock, 2005; Su et al., 2005), the expressions for velocity and dispersion are written in degenerate form. New space and time variables are introduced at the different stages through different transformations. It enables to reduce the variable coefficients into constant coefficients. So a much simpler but more viable Laplace transformation technique is used to get the analytical solution. Such solutions will be very useful in validating a numerical solution of a more general dispersion problem by infinite element technique (Zhao and Valliappan, 1994; Zhao, 2009), and other advanced numerical techniques (Herrera et al., 2010). The solution is illustrated in a way to demonstrate the solute transport along the lateral direction. It has been found significant even in very low velocity and dispersivity along this direction compared to the respective longitudinal parts. It shows that a two-dimensional model is more useful than a one-dimensional model. Also solutions for different combinations of unsteadiness of both the coefficients may be obtained as particular cases from the one obtained in the present study. More solutions may be illustrated for any value of n in the theory of Freeze and Cherry (1979), due to the unsteadiness.

2. Mathematical formulation and analytical solution

Let solute particles of a pollutant enter a medium of the environment of air or soil or water, at a site, continuously at a uniform rate up to a certain time period, and just after that, let it become zero. In other words, the source of pollution is a stationary uniform pulse-type point source. For example, smoke coming out from a chimney of a factory; particulate particles coming out of a volcano; the sewage outlet of a municipal area or effluent outlet of a factory or industry in a surface water medium; infiltrations of wastes from garbage disposal sites, septic tanks, mines, discharge from surface water bodies polluted due to industrial and municipal influents, and reaching the ground water level, particularly with rainwater. Let the solute particles be transported down the flow-stream due to diffusion and convection, in a horizontal plane at the site of the input concentration. Let the two perpendicular directions, both of semi-infinite extents, in the horizontal plane be longitudinal direction ($0 \leqslant x < \infty$) and lateral direction ($0 \leqslant y < \infty$). Let the velocity components of the flow field at a position (x, y) in the horizontal plane be $u(x, t)$ and $v(y, t)$ along the two directions, respectively. Both will satisfy the Darcy law in case the medium is porous or laminar conditions of the flow in case the medium is not porous. Further let $D_x(x, t)$ and $D_y(y, t)$ be longitudinal and lateral components of the solute dispersivity parameter at the same position, respectively. The linear advection–diffusion partial differential equation in two-dimensional horizontal isotropic plane medium in general form may be written as follows:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left[D_x(x,t)\frac{\partial c}{\partial x} - u(x,t)c\right] + \frac{\partial}{\partial y}\left[D_y(y,t)\frac{\partial c}{\partial y} - v(y,t)c\right], \quad (1)$$

where c is the solute concentration of the pollutant, transporting along the flow field through the medium at a position (x, y) at time t.

To solve advection–diffusion Eq. (1) analytically, a set of initial and boundary conditions is assumed. Initially the semi-infinite medium is considered solute free. The source of the pollution is considered to be a uniform pulse-type. The position where the source introduces solute particles in the medium is assumed the origin of the horizontal plane. Let time of elimination of the point source be t_0. Flux type homogeneous conditions are assumed at the far ends of the medium, along both the directions. Thus the initial condition, input condition and the other boundary condition are

$$c(x,y,t) = 0; \quad x \geqslant 0,\ y \geqslant 0,\ t = 0, \quad (2)$$

$$c(x,y,t) = \begin{cases} C_0; & 0 < t \leqslant t_0 \\ 0\ ; & t > t_0 \end{cases}, \quad x = 0,\ y = 0, \quad (3)$$

$$\frac{\partial c}{\partial x} = 0,\ \frac{\partial c}{\partial y} = 0; \quad x \to \infty,\ y \to \infty,\ t \geqslant 0, \quad (4)$$

where C_0 is the reference concentration representing the input concentration released uniformly from the source.

The medium is considered heterogeneous. As a result the velocity of the flow field is considered a spatially dependent function in both the directions. Along each of the two perpendicular directions, one such function is linearly interpolated in terms of respective space variable in a finite region, in which concentration values are evaluated. Each function accounts for a small increase in velocity across the finite region. Further the velocity is also considered temporally dependent. This dependence is considered of same nature in both the longitudinal and lateral directions. Thus the expressions for velocity components are written in the degenerate form as

$$u(x,t) = u_0 f_1(mt)(1 + ax) \quad \text{and} \quad v(y,t) = v_0 f_1(mt)(1 + by), \quad (5)$$