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A new Iterative Alternating Direction Implicit (IADI) algorithm for multi-dimensional saturated–unsaturated flow

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SUMMARY

Numerical simulation of saturated–unsaturated subsurface flows is widely used in many branches of science and engineering, and rapid developments in computer technology have enabled not only one-dimensional but also multi-dimensional simulations on a personal computer. However, a multi-dimensional subsurface flow simulation still incurs heavy load on computational resources, particularly for simulations in wide regions with long periods. An Iterative Alternating Direction Implicit (IADI) scheme has certain advantages in terms of computational cost and algorithmic simplicity. However, it is barely used at present because it occasionally incurs numerical instabilities and convergence difficulties. Another reason is that three-dimensional simulations cannot be performed by the original IADI scheme. This study has proposed an advanced IADI algorithm for solving the saturated–unsaturated flow equation; this advanced scheme is more numerically stable than the original IADI scheme and can be used for three-dimensional simulations. The performance of the proposed scheme was assessed through test simulations. In all the simulations, the new method was shown to be faster than the fully implicit scheme linearized by the modified Picard iteration method while still yielding very similar results.

1. Introduction

Numerical simulation of a saturated–unsaturated subsurface flows is widely used in many branches of science and engineering, including agricultural engineering, ground water engineering, chemical contaminants tracing, and rainfall–runoff modeling. A variety of numerical models have been proposed on the basis of the finite difference, finite element, and finite volume methods to simulate a saturated–unsaturated flow (e.g. Celia et al., 1990; Clement et al., 1994; Forsyth et al., 1995; Jones and Woodward, 2001; Manzini and Ferraris, 2004; Simunek et al., 1999; Tocci et al., 1997). In particular, methods using finite difference algorithms have demonstrated advantages in terms of the ease of coding and understanding owing to their simplicity compared with the other two methods. Rapid developments in computer technology have made it possible to carry out not only one-dimensional (Dane and Mathis, 1981; Haverkamp and Vauclin, 1981) but also multi-dimensional simulations (Clement et al., 1994; Dogan and Motz, 2005; Simunek et al., 1999; Weeks et al., 2004) using a personal computer. However, a multi-dimensional subsurface flow simulation still requires considerable computer resources, particularly for simulations in wide regions with a relatively fine grid resolution.

The alternating direction implicit (ADI) approach and Iterative ADI (IADI) were very popular in the 1970s for avoiding the solution of large, sparse linear systems arising from the implicit discretization of parabolic partial differential equations in 2D and 3D. The IADI scheme is an iterative adaption of the ADI method, which discretizes the equation into a simultaneous system of difference equations that are solved iteratively. Since then, the method has been rarely used in favor of the preconditioned Krylov subspace iteration and even sparse direct solvers. However, the IADI approach has advantages over Krylov solvers in terms of simplicity and cost (on a per iteration basis) because only tridiagonal linear systems are involved in the procedure of the calculation. Optimal Krylov subspace solvers need preconditioners based on multigrid or domain decomposition, which introduce considerably more programming complexity than the IADI method. Furthermore, the computational cost for tridiagonal linear systems is comparatively cheap and proportional to the problem dimension. This implies that the computational cost of the IADI method is expected to be scalable, while the computational cost of the preconditioned Krylov subspace solvers typically increases faster than the problem dimension does. Therefore, if the IADI algorithm overcomes shortcomings such as the instability and convergence difficulty, which will be discussed in the next paragraph, it could be an attractive

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alternative for simulating a saturated–unsaturated flow in porous media. The objective of this study is to propose an improved IADI algorithm for a multi-dimensional saturated–unsaturated flow.

The study by Rubin (1968) was probably the earliest to simulate a two-dimensional transient groundwater flow using the IADI method. Following this study, the IADI method has been used in several studies (e.g. Cooley, 1971; Parissopoulos and Wheater, 1988; Perrens and Watson, 1977; Weeks et al., 2004) to simulate a two-dimensional saturated–unsaturated flow in porous media. All these models solved the pressure-head-based form of the Richards equation (Richards, 1931). However, Celia et al. (1990) stated that numerical methods using the pressure-head-based form of the Richards equation result in a poor mass balance in the unsaturated zone because of the highly non-linear constitutive relationship between the pressure head and the moisture content. These authors showed that solutions based on a mixed form of the Richards equation are more accurate than those based on the pressure-head-based form and satisfy the mass balance. Moreover, Clement et al. (1994) claims that the IADI scheme is not robust because it incurs numerical instabilities and convergence difficulties in solving two-dimensional non-linear equations. This is one of the reasons for rarely using the IADI technique at present. The other reasons might be that the IADI scheme cannot be used for solving three-dimensional problems and that the finite difference-based scheme is limited for real multi-dimensional applications involving complex geometries. To solve the latter problem, we can consider the use of a coordinate transformation method (e.g. Jie et al., 2004; Kinouchi et al., 1991; Koo and Leap, 1998a,b; Ruhaak et al., 2008) or an adaptively refined grid approach (Li et al., 2000) in combination with the IADI scheme.

In this study, in order to overcome the numerical instabilities and applicability of the three-dimensional simulations of the original IADI method of Rubin (1968), we derived a new equation from the ADI method of Douglas and Rachford (1956). The newly derived equation can be applied to two- and three-dimensional problems and shows improved stability. To evaluate the proposed method, five test simulations were conducted, and the results were compared with those of the original IADI scheme and the fully implicit scheme linearized by the modified Picard iteration method.

2. Theory

The mixed form of the Richards equation is generally considered to have advantages in terms of the mass balance and the convergence behavior, which is written as

$$\frac{\partial\theta}{\partial t}=\nabla\cdot K(\psi)\nabla\psi+\frac{\partial K(\psi)}{\partial z}, \tag{1}$$

where ψ is the pressure head, θ is the volumetric moisture content, K is the hydraulic conductivity, t is the time, and z denotes the vertical dimension, assumed to be positive upwards. Further, it is assumed that appropriate constitutive relationships between θ and ψ and between ψ and K are available. The source/sink term has been ignored for the sake of simplicity. Because Eq. (1) includes both θ and ψ, it is called the mixed form.

2.1. Picard iterative linearization

The backward Euler scheme is one of the most widely used time approximation for the Richards equation and used in this study. Since the system of the equation is non-linear because of the non-linear dependency of θ on ψ, iterative calculation and linearization are needed. Although several iterative schemes have been proposed (e.g. Bergamaschi and Putti, 1999; Fassino and Manzini, 1998; Kavetski et al., 2002; Paniconi and Putti, 1994), from a practical viewpoint, the Picard method is used in this study because it is simple and exhibits a good performance in many problems (Lehmann and Ackerer, 1998; Paniconi and Putti, 1994). The backward Euler approximation and Picard linearization of the two-dimensional Eq. (1) is written as

$$\frac{\theta^{n+1,m+1}-\theta^{n}}{\Delta t}=\frac{\partial}{\partial x}\left\{K^{n+1,m}\frac{\partial\psi}{\partial x}\Big|^{n+1,m+1}\right\}+\frac{\partial}{\partial z}\left\{K^{n+1,m}\frac{\partial\psi}{\partial z}\Big|^{n+1,m+1}\right\}+\frac{\partial K}{\partial z}\Big|^{n+1,m}, \tag{2}$$

where the superscripts n and m denote the time level and the iteration level, and x denotes the horizontal dimension.

The moisture content at the new time step and a new iteration level ($\theta^{n+1,m+1}$) is replaced with the Taylor series expansion with respect to ψ, about the expansion point $\psi^{n+1,m}$ as follows:

$$\theta^{n+1,m+1}=\theta^{n+1,m}+\frac{d\theta}{d\psi}\Big|^{n+1,m}(\psi^{n+1,m+1}-\psi^{n+1,m})+O(\delta^2). \tag{3}$$

Neglecting the higher-order terms in Eq. (3) and substituting this equation into Eq. (2) gives

$$C^{n+1,m}\frac{\psi^{n+1,m+1}-\psi^{n+1,m}}{\Delta t}+\frac{\theta^{n+1,m}-\theta^{n}}{\Delta t}=\frac{\partial}{\partial x}\left\{K^{n+1,m}\frac{\partial\psi}{\partial x}\Big|^{n+1,m+1}\right\}+\frac{\partial}{\partial z}\left\{K^{n+1,m}\frac{\partial\psi}{\partial z}\Big|^{n+1,m+1}\right\}+\frac{\partial K}{\partial z}\Big|^{n+1,m}, \tag{4}$$

where $C(=d\theta/d\psi)$ is the specific moisture capacity function. A finite difference approximation of Eq. (4) can be written as

$$C_{i,j}^{n+1,m}\frac{\psi_{i,j}^{n+1,m+1}-\psi_{i,j}^{n+1,m}}{\Delta t}+\frac{\theta_{i,j}^{n+1,m}-\theta_{i,j}^{n}}{\Delta t}=\Delta_x(K^{n+1,m}\Delta_x\psi^{n+1,m+1})+\Delta_z(K^{n+1,m}\Delta_z\psi^{n+1,m+1})+\Delta_z(K^{n+1,m}), \tag{5}$$

where

$$\Delta_x(K^{n+1,m}\Delta_x\psi^{n+1,m+1})=\frac{1}{\Delta x^2}K_{i+1/2,j}^{n+1,m}\left(\psi_{i+1,j}^{n+1,m+1}-\psi_{i,j}^{n+1,m+1}\right)-\frac{1}{\Delta x^2}K_{i-1/2,j}^{n+1,m}\left(\psi_{i,j}^{n+1,m+1}-\psi_{i-1,j}^{n+1,m+1}\right),$$

$$\Delta_z(K^{n+1,m}\Delta_z\psi^{n+1,m+1})=\frac{1}{\Delta z^2}K_{i,j+1/2}^{n+1,m}\left(\psi_{i,j+1}^{n+1,m+1}-\psi_{i,j}^{n+1,m+1}\right)-\frac{1}{\Delta z^2}K_{i,j-1/2}^{n+1,m}\left(\psi_{i,j}^{n+1,m+1}-\psi_{i,j-1}^{n+1,m+1}\right),$$

$$\Delta_z(K^{n+1,m})=\frac{K_{i,j+1/2}^{n+1,m}-K_{i,j-1/2}^{n+1,m}}{\Delta z}, \tag{6}$$

and subscripts i and j denote the spatial coordinates in the x and z axes, respectively. Eq. (5) represents the same method proposed by Clement et al. (1994), except that this equation ignores the specific storage term. These linearized simultaneous equations are solved using matrix solvers such as the LU decomposition or preconditioned conjugated gradient methods. In this study, LIS (a library of iterative solvers for linear systems), developed by Kotakemori et al. (2005), was used for solving the linear equations. LIS provides several preconditioners and iterative solvers for linear systems. Conducting the test simulations, we selected a pair of SSOR preconditioner and biconjugate gradient stabilized (BICGSTAB) methods, which was shown to be faster and more stable than the other pairs provided by the LIS library. The pressure head at the (n+1)th time level and the (m+1)th Picard iteration level were obtained solving