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Hantush Well Function revisited

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SUMMARY

In this paper, we comment on some recent numerical and analytical work to evaluate the Hantush Well Function. We correct an expression found in a Comment by Nadarajah [Nadarajah, S., 2007. A comment on numerical evaluation of Theis and Hantush–Jacob well functions. Journal of Hydrology 338, 152–153] to a paper by Prodanoff et al. [Prodanoff, J.A., Mansur, W.J., Mascarenhas, F.C.B., 2006. Numerical evaluation of Theis and Hantush–Jacob well functions. Journal of Hydrology 318, 173–183]. We subsequently derived another analytic representation based on a generalized hypergeometric function in two variables and from the hydrological literature we cite an analytic representation by Hunt [Hunt, B., 1977. Calculation of the leaky aquifer function. Journal of Hydrology 33, 179–183]. We have implemented both representations and compared the results. Using a convergence accelerator Hunt's representation of Hantush Well Function is efficient and accurate. While checking our implementations we found that Bear's table of the Hantush Well Function [Bear, J., 1979. Hydraulics of Groundwater. McGraw-Hill, New York, Tables 8–6] contains a number of typographical errors that are not present in the original table published by Hantush [Hantush, M.S., 1956. Analysis of data from pumping tests in leaky aquifers. Transactions, American Geophysical Union 37, 702–714]. Finally, we offer a very fast approximation with a maximum relative error of 0.0033 for the parameter range in the table given by Bear.

1. Introduction

Hantush's Well Function (Hantush and Jacob, 1955) may well be the most popular formula in hydrogeological practice, which is remarkable for an inconvenient mathematical expression that classifies as a special case of the Generalized Incomplete Gamma Function. Ever since its first appearance hydrogeologists have searched for methods to compute the well function; several methods will be reviewed in the next paragraph. In this paper, we review existing analytic methods and discuss two analytic representations. At the end of the paper we present a very fast approximation, which may be useful in programs that require many evaluations of the function, such as models for time series analysis (Asmuth et al., 2008; http://www.menyanthes.nl). We remark that one may alternatively evaluate the Hantush Well Function by performing a numerical inversion of the Laplace transform (2) or by standard numerical integration, for example using Gaussian quadrature (e.g., the Matlab®-code `quadgk`). Both turn out to be also satisfactory. Other methods to evaluate the Hantush Well Function, sometimes purely numerical, have been published, by e.g., Harris (2008) and Temme (2009).

Besides a proposed numerical integration scheme (Prodanoff et al., 2006) presented a review of earlier results. Nadarajah (2007) commented on Prodanoff et al. (2006) to the effect that there was no longer a need for approximate methods, since a closed-form mathematical expression was available, based on an Appell type generalization of the well-known hypergeometric series that allegedly was offered by standard mathematical software. We found some difficulties in evaluating Nadarajah's solution for various reasons which we discuss in this paper (Section 3). Another representation of the Generalized Incomplete Gamma Function uses also an Appell type series generalization of the hypergeometric series (see (13) in Section 4). We rewrite (13) into (24), a representation expressed in the better known Bessel functions I_j and K_0 and discuss its evaluation in Section 6. In Section 5 we consider the closed-form analytic representation by Hunt (1977), which Prodanoff et al. (2006) erroneously called approximate. Using a convergence accelerator, Hunt's representation of Hantush's Well Function is efficient and accurate, which means that one gets high precision for a relatively small number of terms. While checking our implementations we found that Bear's table of the Hantush Well Function (Bear, 1979, Tables 8–6) contains a number of typographical errors, which are not present in the original table published by Hantush (1956). We discuss the performance of the two representations mentioned above in Section 7. In Section 8 we end our paper with a very fast but stable approximate

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expression that is good enough for engineering practice. It is continuous and has continuous first derivatives with respect to its parameters, which is important when it is to be used in an optimization loop. Appendix A contains a Matlab code for this approximation. Our Matlab code for the method by Hunt is available upon request.

2. Hantush Well Function

The Hantush Well Function is defined as

$$W\left(u,\frac{r}{B}\right)=\int_u^\infty \frac{1}{t}\exp\left(-t-\frac{r^2}{4B^2t}\right)dt. \tag{1}$$

This function was introduced in the field of hydrology by Hantush and Jacob (1955). An application with a number of numerical results in table form was given by Hantush (1956). Bear summarized these results and included a table for this function (Bear, 1979, Tables 8–6). The Hantush Well Function was given by Hunt (1977) as a sum over Iterated Exponential Integrals. A recent survey of methods to evaluate the Hantush Well Function was given by Prodanoff et al. (2006). Also, in the mathematical literature attention has been paid to this function, see Harris (2008) and Temme (2009). The Laplace transform of (1) reads

$$\begin{aligned}\overline{W}\left(s;\frac{r}{B}\right)&=\int_0^\infty W\left(u,\frac{r}{B}\right)\exp(-us)du\\&=2\left(K_0\left(\frac{r}{B}\right)-K_0\left(\sqrt{1+s}\frac{r}{B}\right)\right)/s,\end{aligned} \tag{2}$$

where K_0 is the Modified Bessel Function of the Second Kind, order 0.

3. Result of Nadarajah (2007)

In a recent Comment (Nadarajah, 2007) to the paper by Prodanoff et al. (2006) the author points out that it is possible to express (1) in a closed analytical form as a double sum based on the Appell hypergeometric series of the first kind Φ_1^N. The result of Nadarajah (2007) reads

$$W\left(u,\frac{r}{B}\right)=\begin{cases}K_0(r/B)+I, & 0<u<r/(2B),\\ K_0(r/B)-I, & u\geqslant r/(2B),\end{cases} \tag{3}$$

with

$$\begin{aligned}I=&\sqrt{\frac{2Bu}{r}+\frac{r}{2Bu}-2}\exp\left(-\frac{r}{B}\right)\\&\times\Phi_1^N\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{1}{2}-\frac{Bu}{2r}-\frac{r}{8Bu},u+\frac{r^2}{4B^2u}-\frac{r}{B}\right),\end{aligned}$$

where K_0 is the Modified Bessel Function of the Second Kind, order 0, and where Φ_1^N is defined according to Nadarajah as

$$\Phi_1^N(a,b,c;x,y)=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(a)_{m+n}(b)_n x^m y^n}{(c)_{m+n}m!n!}, \tag{4}$$

with

$$(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}=a(a+1)\cdots(a+k-1),\quad a\neq 0,-1,-2,\ldots.$$

We introduced the notation Φ_1^N in stead of Φ_1 for reasons below. We discuss this result with a few remarks:

1. Nadarajah (2007) calls the function Φ_1^N the Appell hypergeometric series of the first kind. This is not correct. The Appell hypergeometric series of the first kind is commonly denoted by F_1 and defined with one extra parameter as (see e.g., Horn, 1931, p. 383; Gradshteyn and Ryzhik, 1965, 9.180)

$$F_1(a,b,b',c;x,y)=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(c)_{m+n}m!n!}, \tag{5}$$

   whereas the function (4) used by Nadarajah (2007) is one of the other functions in two variables introduced by Horn (1931, pp. 383–384) as a generalization of the well-known hypergeometric function.

2. According to Horn (1931, p. 384) and Erdélyi (1954, p. 384) the correct definition for that function Φ_1 reads (see also Srivastava and Karlsson (1985, p. 25, (16)))

$$\Phi_1(a,b,c;x,y)=\sum_{m=0}^\infty\sum_{n=0}^\infty\frac{(a)_{m+n}(b)_m x^m y^n}{(c)_{m+n}m!n!}, \tag{6}$$

   with the restriction of $|x| < 1$ and $|y| < \infty$.
   Note the different subscript for the term (b) in the numerator in (4) and (6). The definition used by Nadarajah (2007) occurs in the literature by Erdélyi (1953, p. 225, (20)) and Gradshteyn and Ryzhik (1965, p. 1067, (9.261)). This has caused some confusion with respect to the results in which Φ_1 is involved. As can easily be seen, there holds

$$\Phi_1(a,b,c;x,y)=\Phi_1^N(a,b,c;y,x). \tag{7}$$

3. Nadarajah (2007) used a result for some specific integral listed in Prudnikov et al. (1986, (2.3.8.1))

$$\begin{aligned}&\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(x+z)^{-\rho}\exp(-px)dx\\&\quad= B(\alpha,\beta)z^{-\rho}a^{\alpha+\beta-1}\Phi_1(\alpha,\rho,\alpha+\beta;-a/z,ap),\end{aligned} \tag{8}$$

   with $\mathrm{Re}\alpha > 0$, $\mathrm{Re}\beta > 0$, $|\arg(1-\sigma)| < \pi$, and $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.
   The function Φ_1 seems not to be defined in that work. This result (8) is related to a similar integral listed in Erdélyi (1954, 4.3, (24))

$$\begin{aligned}&\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(1-\sigma t)^{-\gamma}\exp(-pt)dt\\&\quad=B(\alpha,\beta)\Phi_1(\alpha,\gamma,\alpha+\beta;\sigma,-p),\end{aligned} \tag{9}$$

   with $\mathrm{Re}\alpha > 0$, $\mathrm{Re}\beta > 0$, $|\arg(1-\sigma)| < \pi$.
   The result (9) is in accordance with (8) (after the scaling $t = x/a$, and the change of parameters $\gamma\to\rho$, $\sigma\to -a/z$, $p\to ap$), except for the minus-sign in the last argument for Φ_1. A careful analytical study reveals that (9) is correct. Gradshteyn and Ryzhik (1965, (3.385)) also gave this result (9), but since (Gradshteyn and Ryzhik, 1965, p. 1067, (9.261)) used the wrong definition for Φ_1, it seems that their result is in error. There exists a correction for that result (Gradshteyn and Ryzhik, 1965, (3.385)) (see http://www.mathtable.com/gr) in the sense that the two last arguments for Φ_1 have to be interchanged. It would have been better to correct the definition of Φ_1 in Gradshteyn and Ryzhik (1965, p. 1067, (9.261)).

4. The final conclusion is that the result given by Nadarajah (2007) for the Hantush Well Function is in error. His formula can be repaired by introducing an extra minus-sign for the last argument $\left(u+\frac{r^2}{4B^2u}-\frac{r}{B}\to-\left(u+\frac{r^2}{4B^2u}-\frac{r}{B}\right)\right)$ and by requiring that the correct definition of Φ_1 will be used (i.e. (6)). Moreover, the function Φ_1 converges only for values $\left|\frac{1}{2}-\frac{Bu}{2r}-\frac{r}{8Bu}\right|<1$. So, the correct results reads

$$W\left(u,\frac{r}{B}\right)=\begin{cases}K_0(r/B)+I, & 0<u<r/(2B),\\ K_0(r/B)-I, & u\geqslant r/(2B),\end{cases} \tag{10}$$

   with